Antilinear deformations of Coxeter groups, an application to Calogero models

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Abstract: We construct complex root spaces remaining invariant under antilinear involutions related to all Coxeter groups. We provide two alternative constructions: One is based on deformations of factors of the Coxeter element and the other based on the deformation of the longest element of the Coxeter group. Motivated by the fact that non-Hermitian Hamiltonians admitting an antilinear symmetry may be used to define consistent quantum mechanical systems with real discrete energy spectra, we subsequently employ our constructions to formulate deformations of Coxeter models remaining invariant under these extended Coxeter groups. We provide explicit and generic solutions for the Schrödinger equation of these models for the eigenenergies and corresponding wavefunctions. A new feature of these novel models is that when compared with the undeformed case their solutions are usually no longer singular for an exchange of an amount of particles less than the dimension of the representation space of the roots. The simultaneous scattering of all particles in the model leads to anyonic exchange factors for processes which have no analogue in the undeformed case.

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1. Introduction

It is by now a widely, although not yet universally, accepted fact that non-Hermitian Hamiltonians admitting an antilinear symmetry may be used to define consistent quantum mechanical systems with real energy spectra. This property can be traced back to Wigner's observation [1] that operators invariant under antilinear transformations possess real eigenvalues when their eigenfunctions also respect this symmetry. Particular examples of such an operator and symmetry are for instance a Hamiltonian and $\mathcal{PT}$-symmetry, i.e. a simultaneous parity transformation $\mathcal{P}$ and time reversal $\mathcal{T}$, respectively. Many aspects of the latter possibility are very much explored at present, see e.g. [2, 3], as part of an activity initiated a bit more than a decade ago [4]. Tracing back more than fifty years in the mathematical literature are the closely related and often synonymously used concepts of quasi-Hermiticity [5, 6, 7] and pseudo-Hermiticity [8, 9, 10].

Here we will mainly explore the possibilities arising from antilinear symmetries in general, which are usually realized directly on the dynamical variables in case of quantum mechanical models or on the fields in case of quantum field theories. There exist many models formulated generically in terms of root systems in which the dynamical variables or fields lie in the dual space of the roots, such as Calogero-Moser-Sutherland models, e.g. [11] or Toda field theories, e.g. [12, 13], respectively. Since it is usually quite difficult to identify the symmetry on the level of the variables or fields in a controlled manner, the natural question arises whether it is possible to have a more systematic construction and realize the symmetry directly on the level of the roots, ideally in a completely generic way that is irrespective of a specific root system and also independently of a particular representation for the roots.

The first part of the manuscript, i.e. section 2, is devoted to the development of the mathematical structure and framework where we will provide generic deformations of root systems invariant under Coxeter transformations, crystallographic as well as non-crystallographic ones. Our starting point is to identify involutory maps inside the Coxeter groups and deform them in such a way that they become antilinear. Possible candidates are the Weyl reflections associated to each simple root, the factors of the Coxeter element consisting of commuting Weyl reflections and the longest element. For each of these options we set up a system of constraints for the transformation matrix which maps simple roots into their complex deformations. Subsequently we solve these sets of constraints on a case–by-case basis for all Coxeter groups. Exploiting the fact that some Coxeter groups are embedded into larger ones we also construct additional complex solutions by means of the so-called folding procedure.

The second part of the manuscript is devoted to the application of our constructions to the formulation and study of new models of Calogero type. So far three non-equivalent methods have been explored to deform Calogero models form real to complex systems: i) to add a $\mathcal{PT}$-invariant term to the Hamiltonian [14, 15], ii) to deform the real Calogero model [16, 17, 18] or iii) by constraining or deforming real field equations, like the Boussinesq equation [19], such that the poles of their solutions will give rise to complex Calogero particles. With regard to i) it was shown in [15] that for the terms added so far, the "new" models simply correspond to the original ones with shifted momenta and re-defined coupling constants. The second possibility ii) was first explored in [16], where the deformation was carried out on the level of the dynamical variables for an explicit representation of the A_2-root system. In [18] it was shown that these deformations could be understood generically in the dual space, i.e. directly on the A_2-root space. A construction for the G_2-root spaces and the corresponding Calogero models was also provided in [18], whereas the B_2-case was reported in [19]. It is this construction we generalize to all Coxeter groups in this manuscript, albeit it turns out that the original suggestion based on the deformation of the individual Weyl reflections is too restrictive and may only be carried out for groups of rank 2. Instead this construction needs to be viewed as a special case of a construction based on the deformation factors of the Coxeter element consisting of commuting Weyl reflections. For the antilinearly deformed Calogero Hamiltonian we derive some solutions for the Schrödinger equation for the eigenenergies and wavefunctions. Our solutions are

generalizations of previously constructed ones in the sense that they are formulated in terms of general roots, that is representation independent and irrespective of a specific Coxeter group. Our derivation of the solutions is based on some general identities which we present in appendix A, together with some evidence of their validity. For self-consistency and easy reference we present some case-by-case data for Coxeter groups in appendix B. Possibility iii) is not yet formulated in a generic way in terms of root systems and we will therefore not comment on it here.

2. Root spaces invariant under antilinear involutions

Before considering concrete physical models we will first provide the general mathematical framework, which may also be applied to a different physical setting than the one considered here. Our main aim in this section is to construct complex extended root systems $\tilde{\Delta}(\varepsilon)$ which remain invariant under a newly defined antilinear involutory map. Our starting point is to deform the real roots $\alpha_i \in \Delta \subset \mathbb{R}^n$ and seek to represent them in a complex space depending on some deformation parameter $\varepsilon \in \mathbb{R}$ as $\tilde{\alpha}_i(\varepsilon) \in \tilde{\Delta}(\varepsilon) \subset \mathbb{R}^n \oplus \imath\mathbb{R}^n$. For this purpose we define a linear deformation map

$$\delta: \ \Delta \to \tilde{\Delta}(\varepsilon), \tag{2.1}$$

relating simple roots α and deformed simple roots $\tilde{\alpha}$ as

$$\alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha, \tag{2.2}$$

with the property that the new root space $\tilde{\Delta}$ remains invariant under some antilinear involutory map ω, i.e. $\omega : \tilde{\alpha} = \mu_1\alpha_1 + \mu_2\alpha_2 \mapsto \mu_1^*\omega\alpha_1 + \mu_2^*\omega\alpha_2$ for μ_1, $\mu_2 \in \mathbb{C}$, $\omega^2 = \mathbb{I}$ and $\omega : \tilde{\Delta} \to \tilde{\Delta}$. To achieve this there are clearly various possibilities conceivable. Here we make similar, albeit less constraining, demands as in [18] allowing for a generalization to root spaces invariant under complex extended Coxeter groups $\mathcal{W}$ related to all groups.

2.1 $\mathcal{PT}$-symmetrically deformed Coxeter group factors

We wish to maintain the property that the entire deformed root space $\tilde{\Delta}(\varepsilon)$ can be generated analogously to the undeformed one Δ, namely by some consecutive action of a deformed version of a Coxeter element $\sigma \in \mathcal{W}$ on simple roots. The latter are build up from a product of ℓ simple Weyl reflections

$$\sigma_i(x) := x - 2\frac{x \cdot \alpha_i}{\alpha_i^2}\alpha_i, \qquad \text{with} \quad 1 \leq i \leq \ell \equiv \operatorname{rank}\mathcal{W}, \tag{2.3}$$

that is

$$\sigma = \prod_{i=1}^{\ell} \sigma_i. \tag{2.4}$$

Since reflections do in general not commute, a Coxeter element is only defined up to conjugation and therefore not unique. One way to fix ones conventions is achieved by associating values $c_i = \pm 1$ to the vertices of the Coxeter graphs, in such a way that no two

vertices with the same values are linked together. Consequently the simple roots associated to the vertices split into two disjoints sets, say $V_{\pm}$, such that the Coxeter element can be defined uniquely as

$$\sigma := \sigma_{-}\sigma_{+}, \qquad \text{with } \sigma_{\pm} := \prod_{i \in V_{\pm}} \sigma_i, \tag{2.5}$$

see e.g. [20, 21, 22, 23, 24, 25] for more details. Since all elements in the same set commute, i.e. $[\sigma_i, \sigma_j] = 0$ for $i, j \in V_{+}$ or $i, j \in V_{-}$, the only ambiguity left at this stage is the ordering between the σ_{+} and σ_{-}.

In a general sense we explore here the possibility to identify the antilinear map ω with a $\mathcal{PT}$-symmetry. In this concrete setting the first attempts to pursue this idea to construct complex extended root spaces were made in [18], where the authors used the fact that $\sigma_i^2 = \mathbb{I}$ and identified the Weyl reflections as parity transformations across all hyperplanes separating the Weyl chambers in the rootspace. All these reflections were consistently deformable for $\mathcal{W} = A_2, G_2$ [18] and $\mathcal{W} = B_2$ [19]. For groups with higher rank this amounts to a large number of constraints, which are difficult to solve and might not even have a solution, as we shall see indeed in section 2.3. Nonetheless, for the applications we have in mind, i.e. to guarantee the reality of the spectra of some physical operators as outlined above, one single deformed involutory map is in fact sufficient.

Proceeding in this spirit with Weyl reflections leaves the question of which one to choose as none is particularly distinct. However, there are some very distinguished involutory elements contained in $\mathcal{W}$ of different type, such as the aforementioned (2.5) two factors of the Coxeter element σ_{-} and σ_{+}, both satisfying $\sigma_{-}^2 = \sigma_{+}^2 = \mathbb{I}$. We identify them here as parity transformations and employ them to define two $\mathcal{PT}$-type operators in two alternative ways

$$\sigma_{\pm}^{\varepsilon} := \theta_{\varepsilon}\sigma_{\pm}\theta_{\varepsilon}^{-1} = \tau\sigma_{\pm} : \quad \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon), \tag{2.6}$$

where τ mimics the time-reversal simply acting as a complex conjugation. The operator θ_{ε} constitutes a realization of the deformation map δ in (2.1) relating deformed and undeformed roots as specified in (2.2). The deformed Coxeter element is then naturally defined as

$$\sigma_{\varepsilon} := \theta_{\varepsilon}\sigma\theta_{\varepsilon}^{-1} = \sigma_{-}^{\varepsilon}\sigma_{+}^{\varepsilon} = \tau\sigma_{-}\tau\sigma_{+} = \tau^2\sigma_{-}\sigma_{+} = \sigma : \quad \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon), \tag{2.7}$$

i.e. it acts on $\tilde{\Delta}(\varepsilon)$ in the same way as σ on Δ. This means that the Coxeter transformation and the deformation map commute

$$[\sigma, \theta_{\varepsilon}] = 0. \tag{2.8}$$

Notice that from (2.8) follows that one equation in (2.6) implies the other, i.e. the deformation of σ_{+} yields the deformation of σ_{-} and vice versa. The entire deformed root space $\tilde{\Delta}(\varepsilon)$ can be constructed in analogy to the undeformed case by defining the quantity $\tilde{\gamma}_i = c_i\tilde{\alpha}_i$, which serves as a representant for the deformed Coxeter orbit

$$\Omega_i^{\varepsilon} := \left\{ \tilde{\gamma}_i, \sigma_{\varepsilon}\tilde{\gamma}_i, \sigma_{\varepsilon}^2\tilde{\gamma}_i, \ldots, \sigma_{\varepsilon}^{h-1}\tilde{\gamma}_i \right\} = \theta_{\varepsilon}\Omega_i, \tag{2.9}$$

such that

$$\tilde{\Delta}(\varepsilon) := \bigcup_{i=1}^{\ell} \Omega_i^{\varepsilon} = \theta_\varepsilon \Delta(\varepsilon). \tag{2.10}$$

Note it is not enough to act just on the $\tilde{\alpha}_i$ to generate the entire root space, but dressing them with the colour value will be sufficient [20, 21, 22, 23, 24, 25]. Evidently when defining $\tilde{\Delta}(\varepsilon)$ in this way it will remain invariant under the action of the (deformed) Coxeter element $\sigma_\varepsilon : \tilde{\Delta}(\varepsilon) \to \tilde{\Delta}(\varepsilon)$ and paramount to our intentions the deformed root spaces are $\mathcal{PT}$-symmetric, that is invariant with respect to the action of our map defined in (2.6)

$$\sigma_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon \sigma_\pm \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon \sigma_\pm \Delta(\varepsilon) = \theta_\varepsilon \Delta(\varepsilon) = \tilde{\Delta}(\varepsilon). \tag{2.11}$$

This observation suggests to demand a one-to-one relation between the individual roots, such that $\tilde{\Delta}(\varepsilon)$ is isomorphic to Δ. This is guaranteed with the limit

$$\lim_{\varepsilon \to 0} \tilde{\alpha}_i(\varepsilon) = \alpha_i, \tag{2.12}$$

and therefore we have the reduction $\lim_{\varepsilon \to 0} \tilde{\Delta}(\varepsilon) = \Delta$, i.e. $\lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}$.

In principle, provided θ_ε can be constructed, this will allow us already to formulate new $\mathcal{PT}$-symmetric physical models based on roots by means of the deformation map $\delta : \alpha \mapsto \tilde{\alpha}(\varepsilon)$. However, the number of free parameters is still very large and it is natural to impose further constraints. Motivated by the physical applications we have in mind, we would like the kinetic energy term and possibly other terms in the models to remain invariant under the deformation. This will be guaranteed when we demand the invariance of the inner products in the corresponding root spaces

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j. \tag{2.13}$$

This means θ_ε is an isometry and we demand therefore

$$\theta_\varepsilon^* = \theta_\varepsilon^{-1} \qquad \text{and} \qquad \det \theta_\varepsilon = \pm 1. \tag{2.14}$$

In summary, it turns out that given σ_+ or σ_- for a particular Coxeter group $\mathcal{W}$, we can construct their $\mathcal{PT}$-symmetric, or better antilinear, deformations by solving the constraints (2.6), (2.8), (2.14) and (2.12) that is

$$\theta_\varepsilon^* \sigma_\pm = \sigma_\pm \theta_\varepsilon, \quad [\sigma, \theta_\varepsilon] = 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}, \tag{2.15}$$

for θ_ε. Up to a certain point we demonstrate this now for all Coxeter groups.

In the light of the fact that the Coxeter element commutes with θ_ε and the last relation in (2.15), we make the Ansatz

$$\theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon) \sigma^k, \qquad \text{with } \lim_{\varepsilon \to 0} c_k(\varepsilon) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}, \ c_k(\varepsilon) \in \mathbb{C}. \tag{2.16}$$

Next we try to satisfy the first relation in (2.15). Using the relations $\sigma_- \sigma^{-1} = \sigma\sigma_-$ and $\sigma^h = 1$, we obtain with (2.16)

$$\theta_\varepsilon^* \sigma_- = \sum_{k=0}^{h-1} c_k^*(\varepsilon)\sigma^k \sigma_- = \sum_{k=0}^{h-1} c_k^*(\varepsilon)\sigma_- \sigma^{h-k}, \tag{2.17}$$

which equals

$$\sigma_- \theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon)\sigma_- \sigma^k, \tag{2.18}$$

when

$$c_{h-k}(\varepsilon) = c_k^*(\varepsilon). \tag{2.19}$$

As we expect from the comment after (2.8), the constraint $\theta_\varepsilon^* \sigma_+ = \sigma_+ \theta_\varepsilon$ yields the same relation (2.19) where in the derivation we have to use, however, $\sigma_+ \sigma = \sigma^{-1}\sigma_+$ instead. Since $c_h(\varepsilon) = c_0(\varepsilon)$ the equality (2.19) implies that $c_0(\varepsilon) =: r_0(\varepsilon) \in \mathbb{R}$ and furthermore we deduce that $c_{h/2}(\varepsilon) =: r_{h/2}(\varepsilon) \in \mathbb{R}$ when h is even. Furthermore, we may take the $c_k(\varepsilon)$ to be of the form $c_k(\varepsilon) = \imath r_k(\varepsilon)$. Mostly it turns out that $r_k(\varepsilon) \in \mathbb{R}$, but we will not assume this from the start as we do not wish to exclude possible solutions. Therefore we can write

$$\theta_\varepsilon = \begin{cases} r_0(\varepsilon)\mathbb{I} + \imath \sum\limits_{k=1}^{(h-1)/2} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & \text{for } h \text{ odd}, \\ r_0(\varepsilon)\mathbb{I} + r_{h/2}(\varepsilon)\sigma^{h/2} + \imath \sum\limits_{k=1}^{h/2-1} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & \text{for } h \text{ even}. \end{cases} \tag{2.20}$$

We can easily diagonalize θ_ε by recalling [21] the eigenvalue equation for the Coxeter element

$$\sigma v_n = e^{2\pi\imath s_n/h} v_n, \tag{2.21}$$

with s_n being the exponents of a particular Coxeter group $\mathcal{W}$, see appendix B for explicit values. Defining the matrix $\vartheta = \{v_1, v_2, \ldots, v_\ell\}$, we diagonalize the Coxeter element simply as $\sigma = \vartheta\hat{\sigma}\vartheta^{-1}$ with $\hat{\sigma}_{nn} = e^{2\pi\imath s_n/h}$, such that the deformation matrix diagonalizes as

$$\theta_\varepsilon = \vartheta\hat{\theta}_\varepsilon\vartheta^{-1}, \tag{2.22}$$

with eigenvalues

$$(\hat{\theta}_\varepsilon)_{nn} = \begin{cases} r_0(\varepsilon) - 2\sum\limits_{k=1}^{(h-1)/2} r_k(\varepsilon)\sin\left(\frac{2\pi k}{h}s_n\right) & \text{for } h \text{ odd}, \\ r_0(\varepsilon) + (-1)^{s_n} r_{h/2}(\varepsilon) - 2\sum\limits_{k=1}^{h/2-1} r_k(\varepsilon)\sin\left(\frac{2\pi k}{h}s_n\right) & \text{for } h \text{ even}. \end{cases} \tag{2.23}$$

This means that the constraint $\det\theta_\varepsilon = \pm 1$ in (2.15) is equivalent to $\det\hat{\theta}_\varepsilon = \pm 1$ and therefore

$$\begin{aligned} \pm 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) - 2\sum_{k=1}^{(h-1)/2} r_k(\varepsilon)\sin\left(\tfrac{2\pi k}{h}s_n\right)\right] && \text{for } h \text{ odd}, \\ \pm 1 &= \prod_{n=1}^{\ell}\left[r_0(\varepsilon) + (-1)^n r_{h/2}(\varepsilon) - 2\sum_{k=1}^{h/2-1} r_k(\varepsilon)\sin\left(\tfrac{2\pi k}{h}s_n\right)\right] && \text{for } h \text{ even}. \end{aligned} \tag{2.24}$$

Next we implement the third relation in (2.15), which, using (2.22), corresponds to the ℓ equations

$$\vartheta^{-1}\vartheta^{*}\hat{\theta}_{\varepsilon}(\vartheta^{*})^{-1}\vartheta = \hat{\theta}_{\varepsilon}^{-1}. \tag{2.25}$$

What is left is to find are the $(h-1)/2$ or $h/2+1$ unknown functions $r_i(\varepsilon)$ when h is odd or even, respectively, from the $\ell+1$ equations (2.24) and (2.25). We carry out this task case-by-case for specific Coxter groups in section 2.4.

2.2 $\mathcal{CT}$-symmetrically deformed longest element

Intuitively it would be more natural to have just one deformed involutory map from the start instead of two. In fact there exist one very distinct involution in $\mathcal{W}$, called the longest element. The length of an element in the Coxeter group $\mathcal{W}$ is defined as the smallest amount of simple Weyl reflections σ_i needed to express that element, see e.g. [21]. Since Coxeter groups are finite, there exists an element in $\mathcal{W}$ of maximal length, i.e. the longest element, which we denote as w_0. The length of this element equals the number of positive roots $h\ell$, with h being the Coxeter number of $\mathcal{W}$. The map w_0 is involutive, mapping the set of positive roots $\Delta_+ \subset \mathbb{R}^n$ to negative ones $\Delta_- \subset \mathbb{R}^n$ and vice versa

$$w_0: \quad \Delta_\pm \rightarrow \Delta_\mp, \tag{2.26}$$

where $w_0^2 = \mathbb{I}$. Two specific simple roots, say α_i and $\alpha_{\bar{\imath}}$, are linearly related by w_0 as

$$\alpha_i \mapsto -\alpha_{\bar{\imath}} = (w_0\alpha)_i. \tag{2.27}$$

Here we have borrowed the notation from the context of affine Toda field theories, where it was found [25] that the longest element serves as charge conjugation operator $\mathcal{C}$, mapping a particle of type i to its anti-particle $\bar{\imath}$. From a more mathematical perspective this map is a particular symmetry of the Dynkin diagrams, see e.g. [13].

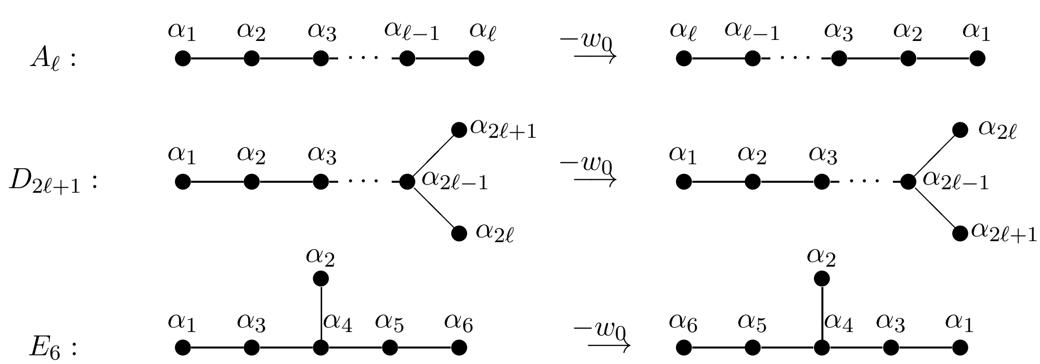

Figure 1: The action of $-w_0$ on the Dynkin diagrams.

The longest element admits a concrete realization in terms of products of Coxeter transformations σ. The unique longest element can be expressed as [25]

$$w_0 = \begin{cases} \sigma^{h/2} & \text{for } h \text{ even}, \\ \sigma_+\sigma^{(h-1)/2} & \text{for } h \text{ odd}. \end{cases} \tag{2.28}$$

For the individual algebras the roots $\alpha_{\bar{\imath}}$ in (2.27) are calculated directly or identified from the symmetries of the Dynkin diagrams [13] as

$$\begin{aligned} A_\ell &: \alpha_{\bar{\imath}} = \alpha_{\ell+1-i}, \\ D_\ell &: \begin{cases} \alpha_{\bar{\imath}} = \alpha_i \quad \text{for } 1 \leq i \leq \ell, & \text{when } \ell \text{ even} \\ \alpha_{\bar{\imath}} = \alpha_i \quad \text{for } 1 \leq i \leq \ell-2,\ \alpha_{\bar{\ell}} = \alpha_{\ell-1}, & \text{when } \ell \text{ odd}, \end{cases} \\ E_6 &: \alpha_{\bar{1}} = \alpha_6, \alpha_{\bar{3}} = \alpha_5, \alpha_{\bar{2}} = \alpha_2, \alpha_{\bar{4}} = \alpha_4, \\ B_\ell, C_\ell, E_7, E_8, F_4, G_2 &: \alpha_{\bar{\imath}} = \alpha_i. \end{aligned} \tag{2.29}$$

Defining then a $\mathcal{CT}$-operator in analogy to (2.6) in two alternative ways, we have

$$w_0^\varepsilon = \theta_\varepsilon w_0 \theta_\varepsilon^{-1} = \tau w_0. \tag{2.30}$$

When $[\sigma, \theta_\varepsilon] = 0$ this equation has no solution for even h, since $w_0^\varepsilon = \theta_\varepsilon \sigma^{h/2} \theta_\varepsilon^{-1} = \sigma^{h/2} = \tau \sigma^{h/2}$, which is evidently a contradiction. Whereas for odd h the realization (2.28) in (2.30) yields $\theta_\varepsilon \sigma_+ \sigma^{(h-1)/2} \theta_\varepsilon^{-1} = \theta_\varepsilon \sigma_+ \theta_\varepsilon^{-1} \sigma^{(h-1)/2} = \tau \sigma_+ \sigma^{(h-1)/2}$, which equals (2.6) when canceling $\sigma^{(h-1)/2}$, such that this case is equivalent to the one described in the previous subsection. This means in order to obtain a new solution from (2.30) we need to assume $[\sigma, \theta_\varepsilon] \neq 0$.

This fact implies immediately that we have now two options to construct the remaining nonsimple roots. We may either define in complete analogy to (2.9) and (2.10) a root space which remains invariant under the action of the deformed Coxeter transformation. This root space is then also $\mathcal{CT}$-symmetric

$$w_0^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon w_0 \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon w_0 \Delta(\varepsilon) = \theta_\varepsilon \Delta(\varepsilon) = \tilde{\Delta}(\varepsilon). \tag{2.31}$$

Alternatively we could also define

$$\hat{\Omega}_i^\varepsilon := \left\{ \tilde{\gamma}_i, \sigma \tilde{\gamma}_i, \sigma^2 \tilde{\gamma}_i, \ldots, \sigma^{h-1} \tilde{\gamma}_i \right\} \tag{2.32}$$

and the entire root space as $\tilde{\Delta}(\varepsilon) := \bigcup_{i=1}^{\ell} \hat{\Omega}_i^\varepsilon$. However, this root space will only remain invariant under the action of σ instead of σ^ε and in addition it will not be $\mathcal{CT}$-symmetric. This definition is therefore unsuitable for our purposes here.

Using the two definitions in (2.30) leads on one hand to

$$w_0^\varepsilon \tilde{\alpha} = \theta_\varepsilon w_0 \theta_\varepsilon^{-1} \theta_\varepsilon \alpha = \theta_\varepsilon w_0 \alpha = -\theta_\varepsilon \bar{\alpha}, \tag{2.33}$$

and on the other to

$$w_0^\varepsilon \tilde{\alpha} = \tau w_0 \tilde{\alpha} = -\tau \bar{\tilde{\alpha}} = -\bar{\tilde{\alpha}}^*, \tag{2.34}$$

such that

$$(\theta_\varepsilon)_{ij} = (\theta_\varepsilon^*)_{\bar{\imath}\bar{\jmath}}. \tag{2.35}$$

As in the previous subsection we require the inner products to be preserved (2.15), such that in summary the set of determining equations result to

$$\theta_\varepsilon^* w_0 = w_0 \theta_\varepsilon, \quad [\sigma, \theta_\varepsilon] \neq 0, \quad \theta_\varepsilon^* = \theta_\varepsilon^{-1}, \quad \det \theta_\varepsilon = \pm 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}. \tag{2.36}$$

In this case it is instructive to separate θ_ε into its real and imaginary part $(\theta_\varepsilon)_{ij} = R_i^j(\varepsilon) + \imath I_i^j(\varepsilon)$ and therefore expand an arbitrary simple deformed root in terms of the ℓ simple roots as

$$\tilde{\alpha}_i(\varepsilon) := \sum_{j=1}^{\ell} \left(R_i^j(\varepsilon)\alpha_j + \imath I_i^j(\varepsilon)\alpha_j \right), \tag{2.37}$$

with $R_i^j(\varepsilon)$ and $I_i^j(\varepsilon)$ being some real valued functions satisfying

$$\lim_{\varepsilon\to 0} R_i^j(\varepsilon) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \qquad \text{and} \qquad \lim_{\varepsilon\to 0} I_i^j(\varepsilon) = 0. \tag{2.38}$$

The relation (2.35) then implies that

$$R_i^j(\varepsilon) = R_{\bar{\imath}}^{\bar{\jmath}}(\varepsilon) \qquad \text{and} \qquad I_i^j(\varepsilon) = -I_{\bar{\imath}}^{\bar{\jmath}}(\varepsilon). \tag{2.39}$$

This means for Coxeter groups in which for all simple roots are self-conjugate $\alpha_i = \alpha_{\bar{\imath}}$ a nontrivial complex $\mathcal{CT}$-symmetric deformation of the longest element can not exist.

2.3 $\mathcal{PT}$-symmetrically deformed Weyl reflections

Yet another possibility would be to identify the parity operator $\mathcal{P}$ with the Weyl reflections σ_i across all hyperplanes separating the Weyl chambers as suggested in [18]. For rank 2 Coxeter groups this construction is identical to the one in section 2.1 and the natural question is whether it can be generalized to higher rank. We present here a simple argument which proves that this is in fact not possible.

Assuming that we can consistently deform at least three Weyl reflections according to

$$\sigma_i^\varepsilon = \theta_\varepsilon \sigma_i \theta_\varepsilon^{-1} = \tau\sigma_i, \qquad \text{for } i = j, k, l, \tag{2.40}$$

it follows that

$$\sigma_j^\varepsilon \sigma_k^\varepsilon \sigma_l^\varepsilon = \theta_\varepsilon \sigma_j \sigma_k \sigma_l \theta_\varepsilon^{-1} = \tau^3 \sigma_j \sigma_k \sigma_l = \tau \sigma_j \sigma_k \sigma_l. \tag{2.41}$$

Demanding that inner products are preserved, we may employ (2.14) and combine it with (2.40) to derive $\sigma_i = \theta_\varepsilon \sigma_i \theta_\varepsilon$. Therefore we have

$$\sigma_j \sigma_k \sigma_l = \theta_\varepsilon \sigma_j \theta_\varepsilon \sigma_k \theta_\varepsilon \sigma_l \theta_\varepsilon = \theta_\varepsilon \sigma_j \sigma_k \sigma_l \theta_\varepsilon, \tag{2.42}$$

which together with (2.41) implies that

$$\tau = \theta_\varepsilon^2. \tag{2.43}$$

As this is impossible to solve this means more than two Weyl reflections can not be consistently $\mathcal{PT}$-deformed in a simultaneous manner.

2.4 Deformed root systems, case-by-case solutions

On a case-by case basis for individual Coxeter groups we will now provide explicit solutions for the set of constraining equations for deformed root systems based on antilinear deformations of the $\sigma_\pm$ as outlined in section 2.1 and where possible also based on deformations of the longest element w_0 as explained in section 2.2.

2.4.1 $\tilde{\Delta}(\varepsilon)$ for A_ℓ

Our convention for the labelling of the roots is depicted in figure 1.

$\tilde{\Delta}(\varepsilon)$ for A_2

As explained after equation (2.30), we should obtain identical deformed root spaces from the two different construction methods in this case as the Coxeter number h is odd .

$\mathcal{CT}$-symmetrically deformed longest element Let us start with the construction of a $\mathcal{CT}$-symmetric deformation of w_0. According to (2.29) in the A_2-case the two simple roots are related to each other by the longest element or in affine Toda particle terminology they are conjugate to each other, i.e. $\bar{1} = 2$. Using the expansion (2.37) and the constraints (2.39) the deformed roots acquire the form

$$\tilde{\alpha}_1 = R_1^1(\varepsilon)\alpha_1 + R_1^2(\varepsilon)\alpha_2 + \imath(I_1^1(\varepsilon)\alpha_1 + I_1^2(\varepsilon)\alpha_2), \tag{2.44}$$

$$\tilde{\alpha}_2 = R_1^2(\varepsilon)\alpha_1 + R_1^1(\varepsilon)\alpha_2 - \imath(I_1^2(\varepsilon)\alpha_1 + I_1^1(\varepsilon)\alpha_2). \tag{2.45}$$

Demanding next that the inner products are preserved (2.13) amounts to three further constraint, such that the four free functions in (2.44), (2.45) are reduced to only one. We obtain the two solutions

$$R_1^2 = 0, \quad I_1^2 = 2I_1^1, \quad \left(R_1^1\right)^2 - \frac{3}{4}\left(I_1^2\right)^2 = 1 \quad \text{and} \quad 1 \leftrightarrow 2. \tag{2.46}$$

The third relation in (2.46) is solved for instance by $R_1^1 = \cosh\varepsilon$, $I_1^2 = 2/\sqrt{3}\sinh\varepsilon$ satisfying also the limiting constraint (2.38) for $\varepsilon \to 0$. Accordingly, the deformed simple roots are

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + \imath\frac{1}{\sqrt{3}}\sinh\varepsilon(\alpha_1 + 2\alpha_2), \tag{2.47}$$

$$\tilde{\alpha}_2 = \cosh\varepsilon\alpha_2 - \imath\frac{1}{\sqrt{3}}\sinh\varepsilon(2\alpha_1 + \alpha_2). \tag{2.48}$$

Different solutions to (2.46) may of course be found. With the representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \sigma = \sigma_1\sigma_2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, w_0 = \sigma_2\sigma = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.49}$$

the deformed longest element results with (2.30) to

$$w_0^\varepsilon = \begin{pmatrix} -\frac{2\imath}{\sqrt{3}}\sinh(2\varepsilon) & -\cosh(2\varepsilon) - \frac{\imath}{\sqrt{3}}\sinh(2\varepsilon) \\ -\cosh(2\varepsilon) + \frac{\imath}{\sqrt{3}}\sinh(2\varepsilon) & \frac{2\imath}{\sqrt{3}}\sinh(2\varepsilon) \end{pmatrix}. \tag{2.50}$$

It is easy to verify that (2.34) is satisfied, namely $w_0^\varepsilon : \tilde{\alpha}_1 \mapsto -\tilde{\alpha}_2^*, \tilde{\alpha}_2 \mapsto -\tilde{\alpha}_1^*$.

$\mathcal{PT}$-symmetrically deformed Coxeter group factors Alternatively we may use the Ansatz (2.16)

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + \imath r_1(\varepsilon)(\sigma - \sigma^2) \tag{2.51}$$

where σ is given in (2.49). The constraint $\det\theta_\varepsilon = 1$, (2.24) with $s_n = n$ and $h = 3$ yields $r_0^2 - 3r_1^2 = 1$ with solutions $r_0 = \cosh\varepsilon, r_1 = -1/\sqrt{3}\sinh\varepsilon$. There are no further

constraints resulting from (2.25) as with $\vartheta = \{(e^{\imath\pi/3}, e^{-\imath\pi/3}), e^{\imath\pi 2/3}, e^{-\imath\pi 2/3})\}$ it is trivially satisfied when $r_0^2 - 3r_1^2 = 1$. Therefore we have

$$\theta_\varepsilon = \cosh\varepsilon \mathbb{I} - \imath\frac{1}{\sqrt{3}}\sinh\varepsilon\left(\sigma - \sigma^2\right). \tag{2.52}$$

With (2.2) we obtain from this exactly the roots in (2.47) and (2.48), thus confirming the expectations announced at the beginning of this subsection.

Note that in this case the constraint even holds for the individual Weyl reflections, i.e. $\sigma_1\theta_\varepsilon = (\theta_\varepsilon\sigma_1)^*$ and $\sigma_2\theta_\varepsilon = (\theta_\varepsilon\sigma_2)^*$ as $\sigma_1 = \sigma_-$ and $\sigma_2 = \sigma_+$. This means we can view this deformation in an alternative way as deformations across every hyperplane in the A_2-root system. The latter was the constraint imposed in [18], which explains that (2.47) and (2.48) are precisely the deformations constructed therein.

The remaining positive nonsimple root is simply $\tilde{\alpha}_1 + \tilde{\alpha}_2$ due to the fact that $\sigma_\varepsilon = \sigma$.

$\tilde{\Delta}(\varepsilon)$ for A_3

Now the Coxeter number is even, such that according to the reasoning after equation (2.30) we expect to obtain two different types of deformed root systems from the two different methods of construction.

$\mathcal{PT}$-symmetrically deformed Coxeter group factors The Ansatz (2.16) reads now

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^3\right) = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - r_2 \\ 2\imath r_1 & r_0 - r_2 + 2\imath r_1 & 2\imath r_1 \\ -\imath r_1 - r_2 & -2\imath r_1 & r_0 - \imath r_1 \end{pmatrix} \tag{2.53}$$

where we represent

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \sigma_+ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.54}$$

$$\sigma_- = \sigma_1\sigma_3 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \sigma = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \vartheta = \begin{pmatrix} 1 & -1 & 1 \\ -(1+\imath) & 0 & \imath - 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.55}$$

The constraints (2.24) and (2.25) yield

$$(r_0 + r_2)\left[(r_0 + r_2)^2 - 4r_1^2\right] = 1, \tag{2.56}$$

$$r_0 - r_2 + 2r_1 = (r_0 - r_2 + 2r_1)(r_0 + r_2), \tag{2.57}$$

$$(r_0 + r_2) = (r_0 - r_2)^2 - 4r_1^2, \tag{2.58}$$

where we used $s_n = n$ and $h = 4$ to derive (2.56). Equations (2.56)-(2.58) are solved for instance by

$$r_0(\varepsilon) = \cosh\varepsilon, \quad r_1(\varepsilon) = \pm\sqrt{\cosh^2\varepsilon - \cosh\varepsilon} \quad \text{and} \quad r_2(\varepsilon) = 1 - \cosh\varepsilon. \tag{2.59}$$

The three simple deformed roots are therefore

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + (\cosh\varepsilon - 1)\alpha_3 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3), \tag{2.60}$$

$$\tilde{\alpha}_2 = (2\cosh\varepsilon - 1)\alpha_2 + 2\imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + \alpha_2 + \alpha_3), \tag{2.61}$$

$$\tilde{\alpha}_3 = \cosh\varepsilon\alpha_3 + (\cosh\varepsilon - 1)\alpha_1 - \imath\sqrt{2}\sqrt{\cosh\varepsilon}\sinh\left(\frac{\varepsilon}{2}\right)(\alpha_1 + 2\alpha_2 + \alpha_3). \tag{2.62}$$

Making use of (2.7) the three remaining positive nonsimple roots are $\tilde{\alpha}_4 := \tilde{\alpha}_1 + \tilde{\alpha}_2$, $\tilde{\alpha}_5 := \tilde{\alpha}_2 + \tilde{\alpha}_3$ and $\tilde{\alpha}_6 := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3$.

$\mathcal{CT}$-symmetrically deformed longest element We obtain an additional non-equivalent solution when $[\sigma, \theta_\varepsilon] \neq 0$ by solving (2.30). For A_3 we read off from (2.29) that $\bar{1} = 3$, $\bar{2} = 2$, such that (2.35) leads to the deformation matrix

$$\theta_\varepsilon = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} = \theta_{22}^* & \theta_{21}^* \\ \theta_{13}^* & \theta_{12}^* & \theta_{11}^* \end{pmatrix}. \tag{2.63}$$

Substituting this into (2.36) yields a set of constraining equations. Assuming θ_{12} to vanish they simplify to

$$\theta_{22} = |\theta_{11}|^2 - |\theta_{13}|^2, \quad \theta_{22}^2 = 1, \quad |\theta_{11}|^2 - \theta_{13}^2 = 1, \tag{2.64}$$

$$\theta_{11}\theta_{21}^* = \theta_{21}(\theta_{22} + \theta_{13}^*), \quad \theta_{11}\operatorname{Re}\theta_{13} = 0. \tag{2.65}$$

Making now only the one further assumption that $\theta_{11} = \cosh\varepsilon$ all remaining entries are fixed by (2.64) and (2.65). We obtain

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon & 0 & \imath\sinh\varepsilon \\ (-\sinh^2\frac{\varepsilon}{2} + \frac{\imath}{2}\sinh\varepsilon) & 1 & (-\sinh^2\frac{\varepsilon}{2} - \frac{\imath}{2}\sinh\varepsilon) \\ -\imath\sinh\varepsilon & 0 & \cosh\varepsilon \end{pmatrix}. \tag{2.66}$$

It is easily verified that the corresponding roots have the desired behaviour under the $\mathcal{CT}$-transformation, namely $\tilde{w}_0(\tilde{\alpha}_1) = -\tilde{\alpha}_3$, $\tilde{w}_0(\tilde{\alpha}_2) = -\tilde{\alpha}_2$. This solution does not correspond to a deformation of $\sigma_\pm$ as now $\theta_\varepsilon^*\sigma_\pm \neq \sigma_\pm\theta_\varepsilon$.

In this case the nonsimple roots can not be constructed from a simple analogy to the undeformed case as $\sigma_\varepsilon \neq \sigma$. Instead we have to act successively with σ_ε on the simple deformed roots. In this way the set of all positive deformed roots results to

$$\tilde{\alpha}_1 = \cosh\varepsilon\alpha_1 + \imath\sinh\varepsilon\alpha_3, \tag{2.67}$$

$$\tilde{\alpha}_2 = \alpha_2 - \sinh^2\frac{\varepsilon}{2}(\alpha_1 + \alpha_3) + \frac{\imath}{2}\sinh\varepsilon(\alpha_1 - \alpha_3), \tag{2.68}$$

$$\tilde{\alpha}_3 = \cosh\varepsilon\alpha_3 - \imath\sinh\varepsilon\alpha_1, \tag{2.69}$$

$$\tilde{\alpha}_4 = \cosh\varepsilon(\alpha_1 + \alpha_2) - \imath\sinh\varepsilon(\alpha_2 + \alpha_3), \tag{2.70}$$

$$\tilde{\alpha}_5 = \cosh\varepsilon(\alpha_2 + \alpha_3) + \imath\sinh\varepsilon(\alpha_1 + \alpha_2), \tag{2.71}$$

$$\tilde{\alpha}_6 = \cosh\varepsilon\alpha_2 + \cosh^2\frac{\varepsilon}{2}(\alpha_1 + \alpha_3) + \frac{\imath}{2}\sinh\varepsilon(\alpha_3 - \alpha_1). \tag{2.72}$$

Notice that the nonsimple roots no are no longer just simple roots added together.

$\tilde{\Delta}(\varepsilon)$ for A_4

Using again the Ansatz (2.16) reads now

$$\theta_\varepsilon = r_0(\varepsilon)\mathbb{I} + \imath r_1(\varepsilon)(\sigma - \sigma^4) + \imath r_2(\varepsilon)(\sigma^2 - \sigma^3), \tag{2.73}$$

with the representation

$$\sigma_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.74}$$

$$\sigma = \sigma_1\sigma_3\sigma_2\sigma_4 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \omega = \sigma_2\sigma_4\sigma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.75}$$

In this case the constraints (2.24) and (2.25) yield

$$r_0^4 - 5r_0^2(r_1^2 + r_2^2) + 5(r_2^2 + r_2 r_1 - r_1^2)^2 = 1, \tag{2.76}$$

$$2r_0^2 + \left(-5 + \sqrt{5}\right) r_1^2 - \left(5 + \sqrt{5}\right) r_2^2 + 4\sqrt{5} r_1 r_2 - 2 = 0, \tag{2.77}$$

$$2r_0 + \sqrt{2\left(5 + \sqrt{5}\right)} r_1 + \sqrt{10 - 2\sqrt{5}} r_2 \neq 0, \tag{2.78}$$

where we used $s_n = n$ and $h = 5$ to obtain (2.76). These equations are solved for instance by

$$r_0(\varepsilon) = \cosh\varepsilon, \quad r_1(\varepsilon) = \kappa_- \sinh\varepsilon, \quad r_2(\varepsilon) = \kappa_+ \sinh\varepsilon, \tag{2.79}$$

with $\kappa_\pm = \frac{1}{5}\sqrt{5 \pm 2\sqrt{5}}$. The deformation matrix results to

$$\theta_\varepsilon = \begin{pmatrix} r_0 - \imath r_1 & -2\imath r_1 & -\imath r_1 - \imath r_2 & -2\imath r_2 \\ 2\imath r_1 & r_0 + 2\imath r_1 + \imath r_2 & 2\imath r_1 + 2\imath r_2 & \imath r_1 + \imath r_2 \\ -\imath r_1 - \imath r_2 & -2\imath r_1 - 2\imath r_2 & r_0 - 2\imath r_1 - \imath r_2 & -2\imath r_1 \\ 2\imath r_2 & \imath r_1 + \imath r_2 & 2\imath r_1 & r_0 + \imath r_1 \end{pmatrix}, \tag{2.80}$$

with all entries specified in (2.79). Notice that in this case we also obtain the deformation of the longest element $w_0\theta_\varepsilon = (\theta_\varepsilon w_0)^*$.

$\tilde{\Delta}(\varepsilon)$ for A_5-A_9

Having been very explicit in our previous examples, it suffices to simply list the solutions for the r_i in order to illustrate the working of the Ansatz (2.16) in the following. We find

$$A_5: \quad r_1 = -r_2 = \pm\frac{1}{\sqrt{3}}\sqrt{r_0^2 - r_0}, \quad r_3 = r_0 - 1, \tag{2.81}$$

$$A_6: \quad r_1 = r_2 = -r_3 = 1/\sqrt{7}\sqrt{r_0^2 - 1}, \tag{2.82}$$

$$A_7: \quad r_1 = r_3 = 0, \quad r_2 = \pm\sqrt{r_0^2 - r_0}, \quad r_4 = r_0 - 1, \tag{2.83}$$

$$A_8: \quad r_1 = -r_2 = -\frac{1}{3} r_3 = r_4 = -\frac{\sqrt{r_0^2 - 1}}{3\sqrt{3}}, \tag{2.84}$$

$$A_9: \quad r_1 = -r_4 = -\kappa_-, \quad r_2 = -r_3 = -\kappa_+, \quad r_5 = r_0 - 1. \tag{2.85}$$

In all cases $r_0 = \cosh\varepsilon$ will guarantee that also the last constraint in (2.15) is satisfied. Based on these data one may try to conjecture closed formulae for the entire A-series.

$\tilde{\Delta}(\varepsilon)$ for A_{4n-1}

For the A_{4n-1}-subseries we succeeded to conjecture a closed formula. Setting in (2.16) all $r_k = 0$, except for $k = 0, n, 2n$, the determinant in (2.24) takes on the simple form

$$\det\theta_\varepsilon = (r_0 + r_{2n})^{2n-1}\left(r_0 - 4r_n^2 - 2r_0 r_{2n} + r_{2n}^2\right)^n, \tag{2.86}$$

which equals one for $r_{2n} = 1 - r_0$ and $r_n = \pm\sqrt{r_0^2 - r_0}$. We have verified up to rank 11 that for these values the expression

$$\theta_\varepsilon = r_0\mathbb{I} + r_{2n}\sigma^{2n} + \imath r_n\left(\sigma^n - \sigma^{-n}\right), \tag{2.87}$$

for the deformation matrix also satisfies the first and fourth constraint in (2.15). Once again $r_0 = \cosh\varepsilon$ is a useful choice to guarantee the validity of the last constraint in (2.15).

2.4.2 $\tilde{\Delta}(\varepsilon)$ for B_ℓ

Our convention for the labelling of the roots is to denote the short simple root by α_ℓ.

$\tilde{\Delta}(\varepsilon)$ for B_2 In this case the Ansatz (2.16) reads

$$\theta_\varepsilon = r_0\mathbb{I} + r_2\sigma^2 + \imath r_1\left(\sigma - \sigma^{-1}\right). \tag{2.88}$$

The first four constraints in (2.15) are satisfied for $r_0 = r_2 \pm\sqrt{1 + 4r_1^2}$, which in turn is conveniently solved for $r_0 = \cosh\varepsilon$, $r_2 = 0$ and $r_1 = 1/2\sinh\varepsilon$, such that simple roots and simple deformed roots are related according to (2.2) by

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon - \imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ \imath\sinh\varepsilon & \cosh\varepsilon + \imath\sinh\varepsilon \end{pmatrix}. \tag{2.89}$$

This solution coincides with the one reported in [19].

$\tilde{\Delta}(\varepsilon)$ for B_3 In this case the Ansatz (2.16)

$$\theta_\varepsilon = r_0\mathbb{I} + r_3\sigma^3 + \imath r_1\left(\sigma - \sigma^{-1}\right) + \imath r_2\left(\sigma^2 - \sigma^{-2}\right), \tag{2.90}$$

is only solving the first four constraints in (2.15) when $r_0 = r_3 - 1$ and $r_1 = -r_2$, which however corresponds to a trivial real solution with $(\theta_\varepsilon)_{ii} = -1$ for $i = 1, 2, 3$.

$\tilde{\Delta}(\varepsilon)$ for B_4 In this case the Ansatz (2.16)

$$\theta_\varepsilon = r_0\mathbb{I} + r_4\sigma^4 + \imath r_1\left(\sigma - \sigma^{-1}\right) + \imath r_2\left(\sigma^2 - \sigma^{-2}\right) + \imath r_3\left(\sigma^3 - \sigma^{-3}\right), \tag{2.91}$$

is solving the first four constraints in (2.15) when $r_0 = r_4 \pm\sqrt{1 + 4r_2^2}$ and $r_1 = -r_3$. We may incorporate the last constraint in (2.15) by solving this with $r_0 = \cosh\varepsilon$, $r_4 = 0$ and $r_2 = 1/2\sinh\varepsilon$, such that the deformation matrix becomes

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon & 0 & -\imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ 0 & \cosh\varepsilon + \imath\sinh\varepsilon & 2\imath\sinh\varepsilon & 2\imath\sinh\varepsilon \\ -\imath\sinh\varepsilon & -2\imath\sinh\varepsilon & \cosh\varepsilon - 2\imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ \imath\sinh\varepsilon & \imath\sinh\varepsilon & \imath\sinh\varepsilon & \cosh\varepsilon + \imath\sinh\varepsilon \end{pmatrix}. \tag{2.92}$$

$\tilde{\Delta}(\varepsilon)$ for B_{2n+1} Supported by the previous examples and supplemented with several more for higher rank not presented here, we conjecture that there are no complex solutions for our constraints in the case of odd rank in the B-series based of the Ansatz (2.16).

$\tilde{\Delta}(\varepsilon)$ for B_{2n} Extrapolation from B_2 and B_4 we conjecture a closed formula for the even rank in the B-series

$$\theta_\varepsilon = r_0 \mathbb{I} + \frac{\imath}{2} r_n \left(\sigma^n - \sigma^{-n} \right), \tag{2.93}$$

for the solution of the first four constraints in (2.15). It is easily seen from (2.24) that the determinant of θ_ε in (2.93) results to

$$\det \theta_\varepsilon = \prod_{k=1}^{n} \left[r_0 - 2 r_n \sin \left(\frac{2\pi n}{4n} s_k \right) \right] = \left(r_0^2 - 4 r_n^2 \right)^n, \tag{2.94}$$

when using the fact that $h = 4n$ and $s_k = 2k - 1$. Choosing $r_0 = \cosh \varepsilon$ and $r_n = 1/2 \sinh \varepsilon$ will then ensure that the last two constraints in (2.15) are also satisfied. It turns out that the remaining equations are solved automatically without any further restrictions. We have verified this on a case-by-case basis up to rank 8.

2.4.3 $\tilde{\Delta}(\varepsilon)$ for C_ℓ

This case can be solved in a completely analogous way to the B_n-series. Equation (2.94) is absolutely identical to B_{2n} and we find that the Ansatz (2.93) together with the relevant r_n also solves the remaining constraints, which we have verified up to rank 8. Once again we did not find any complex solutions up to that order of the rank for C_{2n+1}-series and conjecture also in this case that they do not exist when based on the Ansatz (2.93).

2.4.4 $\tilde{\Delta}(\varepsilon)$ for D_ℓ

Our convention for the labelling of the roots is depicted in figure 1.

$\mathcal{PT}$-symmetrically deformed Coxeter group factors For the odd rank subseries, that is D_{2n+1}, we find a closed formula very similar to the one for A_{4n-1}. This is not surprising given the fact that these two groups are embedded into each other as $D_{2n+1} \hookrightarrow A_{4n-1}$. We find that the deformation matrix of the form

$$\theta_\varepsilon = r_0 \mathbb{I} + r_{2n} \sigma^{2n} + \imath r_n \left(\sigma^n - \sigma^{-n} \right), \tag{2.95}$$

solves the first four constraints in (2.15) with $r_{2n} = 1 - r_0$ and $r_n = \pm\sqrt{r_0^2 - r_0}$. The choice $r_0 = \cosh \varepsilon$ ensures the validity of last constraint in (2.15).

There are no complex solutions for D_{2n} based on the Ansatz (2.16). For instance, considering the Ansatz for D_4 the constraining equations force us to take $r_1 = -r_2$ and $r_3 = r_0 - 1$, which reduces θ_ε to the identity matrix $\mathbb{I}$. Similarly the constraints for the Ansatz (2.16) for D_6 demand that $r_1 = -r_4$, $r_2 = -r_3$ and $r_5 = r_0 - 1$, which reduces θ_ε again to the identity matrix $\mathbb{I}$.

$\mathcal{CT}$-symmetrically deformed longest element For the odd rank subseries we should also be able to construct an alternative solution by solving (2.36). As a special solution valid for the entire subseries we find

$$\theta_\varepsilon = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \hat{\theta}_\varepsilon \end{pmatrix}, \tag{2.96}$$

with

$$\hat{\theta}_\varepsilon = \begin{pmatrix} 1 & (-\sinh^2\frac{\varepsilon}{2} - \frac{\imath}{2}\sinh\varepsilon) & (-\sinh^2\frac{\varepsilon}{2} + \frac{\imath}{2}\sinh\varepsilon) \\ 0 & \cosh\varepsilon & -\imath\sinh\varepsilon \\ 0 & -\imath\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}. \tag{2.97}$$

The solutions (2.95) and (2.96) do not coincide

2.4.5 $\tilde{\Delta}(\varepsilon)$ for E_6

Our convention for the labelling of the roots is depicted in figure 1.

$\mathcal{PT}$-symmetrically deformed Coxeter group factors As we have seen in the previous examples we have usually more parameters at our disposal than we require to solve the constraining equations. Thus instead of finding the most general solution we will be content here to solve (2.16) for some restricted set of values and attempt to solve the constraints in (2.15) for

$$\theta_\varepsilon = r_0\mathbb{I} + \imath r_k\left(\sigma^k - \sigma^{-k}\right). \tag{2.98}$$

Considering (2.24) for this Ansatz yields

$$1 = \prod_{n=1}^{6}\left[r_0 - 2r_k\sin\left(\frac{\pi k}{6}s_n\right)\right] \qquad \text{with } s_n = 1,4,5,7,8,11, \tag{2.99}$$

which reduces to

$$1 = \left(r_0^2 - 3r_k^2\right)^3 \qquad \text{for} \quad k = 2,4. \tag{2.100}$$

It turns out that in both cases the solution $r_k = \pm 1/\sqrt{3}\sqrt{r_0^2-1}$ for (2.100) also solves the first three constraints in (2.15). For the deformation matrix we then obtain for $k=2$

$$\theta_\varepsilon = \begin{pmatrix} r_0 & -2\imath r_2 & 0 & -2\imath r_2 & -2\imath r_2 & -\imath r_2 \\ 2\imath r_2 & r_0+\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 & 2\imath r_2 \\ 0 & 2\imath r_2 & r_0+2\imath r_2 & 4\imath r_2 & 3\imath r_2 & 2\imath r_2 \\ -2\imath r_2 & -2\imath r_2 & -4\imath r_2 & r_0-5\imath r_2 & -4\imath r_2 & -2\imath r_2 \\ 2\imath r_2 & 2\imath r_2 & 3\imath r_2 & 4\imath r_2 & r_0+2\imath r_2 & 0 \\ -\imath r_2 & -2\imath r_2 & -2\imath r_2 & -2\imath r_2 & 0 & r_0 \end{pmatrix}, \tag{2.101}$$

and for $k=4$

$$\theta_\varepsilon = \begin{pmatrix} r_0-\imath r_4 & -2\imath r_4 & -2\imath r_4 & -2\imath r_4 & 0 & 0 \\ 2\imath r_4 & r_0+\imath r_4 & 2\imath r_4 & 2\imath r_4 & 2\imath r_4 & 2\imath r_4 \\ 2\imath r_4 & 2\imath r_4 & r_0+3\imath r_4 & 4\imath r_4 & 2\imath r_4 & 0 \\ -2\imath r_4 & -2\imath r_4 & -4\imath r_4 & r_0-5\imath r_4 & -4\imath r_4 & -2\imath r_4 \\ 0 & 2\imath r_4 & 2\imath r_4 & 4\imath r_4 & r_0+3\imath r_4 & 2\imath r_4 \\ 0 & -2\imath r_4 & 0 & -2\imath r_4 & -2\imath r_4 & r_0-\imath r_4 \end{pmatrix}. \tag{2.102}$$

In each case we may specify further $r_0 = \cosh\varepsilon$ such that $r_k = 1/\sqrt{3}\sinh\varepsilon$ in order to ensure also the right limiting behaviour, i.e. the last constraint in (2.15).

$\mathcal{CT}$-symmetrically deformed longest element We obtain an additional solution by means of the construction laid out in section 2.2. As a particular solution we find

$$\theta_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \theta_\varepsilon^{A_3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.103}$$

with $\theta_\varepsilon^{A_3}$ given in (2.66). This means the fact that the subsystem made from the vertices $3, 4$ and 5 is identical to A_3 also reflects in the solution for the deformation matrix. Clearly this solution is different from (2.101) as well as (2.102).

2.4.6 $\tilde{\Delta}(\varepsilon)$ for E_7

Our convention for the labelling of the roots is the same as for E_6 by linking the additional root α_7 to α_6. There exists no complex solution to (2.15) based on the Ansatz (2.16) with $h = 18$. Together with the explicit representation for the σ we substitute this into the constraints (2.15) and find the unique real solution for the unknown functions $r_0 = 1 + r_5$, $r_1 = -r_4 - r_5 - r_8$, $r_2 = -r_4 - r_5 - r_7$ and $r_3 = -r_6$, which reduced the deformation matrix to the identity operator $\theta_\varepsilon = \mathbb{I}$.

2.4.7 $\tilde{\Delta}(\varepsilon)$ for E_8

Our convention for the labelling of the roots is the same as for E_7 by linking the additional root α_8 to α_7. Also in this case there exists no complex solution to (2.15) based on the Ansatz (2.16) with $h = 30$. When substituted into the constraints (2.15) we find the unique solution $r_0 = 1 + r_5$, $r_1 = -r_5 - r_6 - r_9 - r_{10} - r_{14}$, $r_2 = -2r_5 - r_7 - r_8 - 2r_{10} - r_{13}$, $r_3 = -r_5 - r_7 - r_8 - r_{10} - r_{12}$ and $r_4 = r_5 - r_6 - r_9 + r_{10} - r_{11}$. However, this solution simply corresponds to $\theta_\varepsilon = \mathbb{I}$.

2.4.8 $\tilde{\Delta}(\varepsilon)$ for F_4

Our convention for the labelling of the roots is to denote the long roots by α_1,α_2 and short roots by α_3,α_4 with α_i linked to α_{i+1} for $i = 1, 2, 3$. In the F_4-Ansatz (2.16)

$$\theta_\varepsilon = r_0\mathbb{I} + r_6\sigma^6 + \imath\sum_{k=1}^{5} r_k\left(\sigma^k - \sigma^{-k}\right) \tag{2.104}$$

we have seven unknown functions left. We find two inequivalent solutions for the first four constraints in (2.15), when specifying only two functions, either

$$r_1 = -2r_3 - r_5 \pm \sqrt{(r_0 - r_6)^2 - 1} \qquad \text{and} \qquad r_2 = -r_4 \tag{2.105}$$

or

$$r_1 = -2r_3 - r_5 \qquad \text{and} \qquad r_2 = -r_4 \pm \frac{1}{\sqrt{3}}\sqrt{(r_0 - r_6)^2 - 1}. \tag{2.106}$$

This leaves five functions at our disposal, which we may choose in accordance with the last constraint in (2.15). Taking for instance $r_3 = r_4 = r_5 = r_6 = 0$ and $r_0 = \cosh\varepsilon$ in (2.105) yields

$$\theta_\varepsilon = \begin{pmatrix} \cosh\varepsilon - \imath\sinh\varepsilon & -2\imath\sinh\varepsilon & -2\imath\sinh\varepsilon & 0 \\ 2\imath\sinh\varepsilon & \cosh\varepsilon + 3\imath\sinh\varepsilon & 4\imath\sinh\varepsilon & 2\imath\sinh\varepsilon \\ -\imath\sinh\varepsilon & -2\imath\sinh\varepsilon & \cosh\varepsilon - 3\imath\sinh\varepsilon & -2\imath\sinh\varepsilon \\ 0 & \imath\sinh\varepsilon & 2\imath\sinh\varepsilon & \cosh\varepsilon + \imath\sinh\varepsilon \end{pmatrix}, \qquad (2.107)$$

for the deformation matrix.

2.4.9 $\tilde{\Delta}(\varepsilon)$ for G_2

We label the short root by α_1 and the long root by α_2. As mentioned, this case has been solved before [18], but nonetheless we report it here for completeness and to demonstrate that it fits well into the general framework provided here. The Ansatz (2.16) with $h = 6$ solves the first four constraints (2.15) uniquely with $r_3 = 0$ and $r_0 = \pm\sqrt{1 + 3(r_1 + r_2)^2}$. The choice $r_1 = 1/\sqrt{3}\sinh\varepsilon - r_2$ reproduces the result of [18].

This completes the study of all crystallographic Coxeter groups. We will also consider one noncrystallographic example.

2.4.10 $\tilde{\Delta}(\varepsilon)$ for H_3

We label the long roots by α_1, α_2 and the short root by α_3. In this case there are no complex solutions of the type we are seeking here. Substituting the Ansatz (2.16) with $h = 6$ into the constraints (2.15) leads to the unique solution $r_0 = 1$, $r_5 = 0$ and $r_1 + r_4 = -\phi(r_2 + r_3)$ with ϕ being the golden ratio $\phi = (1 + \sqrt{5})/2$ appearing in the H_3-Cartan matrix. However, this solution simply corresponds to $\theta_\varepsilon = \mathbb{I}$.

2.4.11 Solutions from folding

One deficiency of the above constructions is that in some cases they do not lead to any complex solution for $\tilde{\Delta}$. However, we demonstrate now that in these cases one may still construct higher dimensional solutions by means of the so-called folding procedure, see e.g. [13, 26, 27, 28, 29]. This construction makes use of the fact that some root systems are embedded into larger ones. Identifying roots which are related by the involution (2.29), one obtains a root system associated to a different type of Coxeter group. At the same time we may use the folding procedure for consistency checks.

$B_n \hookrightarrow A_{2n}$ We showed that there exist no complex deformations for the B_{2n-1}-series based on the Ansatz (2.16). However, making use of the embedding $B_n \hookrightarrow A_{2n}$ we demonstrate now that one can construct higher dimensional solutions from the reduction of A_{4n-2} to B_{2n-1}. We illustrate this in detail for the particular case $B_3 \hookrightarrow A_6$. Starting with the solution to the constraints (2.15) for A_6-deformation matrix

$$\theta_\varepsilon = r_0\mathbb{I} + \imath r_1\left(\sigma - \sigma^{-1}\right) + \imath r_2\left(\sigma^2 - \sigma^{-2}\right) + \imath r_3\left(\sigma^3 - \sigma^{-3}\right), \qquad (2.108)$$

with $r_1 = r_2 = -r_3 = 1/\sqrt{7}\cosh\varepsilon$, we employ the explicit form for σ to obtain the simple deformed A_6-roots from (2.2)

$$\begin{aligned}
\tilde{\alpha}_1 &= \cosh\varepsilon\alpha_1 - \imath/\sqrt{7}\sinh\varepsilon(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 - 2\alpha_6), && (2.109)\\
\tilde{\alpha}_2 &= \cosh\varepsilon\alpha_2 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4), && (2.110)\\
\tilde{\alpha}_3 &= \cosh\varepsilon\alpha_3 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6), && (2.111)\\
\tilde{\alpha}_4 &= \cosh\varepsilon\alpha_4 + \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 2\alpha_6), && (2.112)\\
\tilde{\alpha}_5 &= \cosh\varepsilon\alpha_5 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6), && (2.113)\\
\tilde{\alpha}_6 &= \cosh\varepsilon\alpha_6 - \imath/\sqrt{7}\sinh\varepsilon(2\alpha_1 - 2\alpha_3 - 2\alpha_4 - 2\alpha_5 - \alpha_6). && (2.114)
\end{aligned}$$

Following the folding procedure we can now define deformed simple B_3-roots as

$$\begin{aligned}
\tilde{\beta}_1 &= \tilde{\alpha}_1 + \tilde{\alpha}_6 = \cosh\varepsilon(\alpha_1 + \alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[3(\alpha_1 - \alpha_6) + 2(\alpha_2 - \alpha_5)], && (2.115)\\
\tilde{\beta}_2 &= \tilde{\alpha}_2 + \tilde{\alpha}_5 = \cosh\varepsilon(\alpha_2 + \alpha_5) + \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_1 - \alpha_6 + \alpha_3 - \alpha_4) + \alpha_2 - \alpha_5], && (2.116)\\
\tilde{\beta}_3 &= \tilde{\alpha}_3 + \tilde{\alpha}_4 = \cosh\varepsilon(\alpha_1 + \alpha_6) - \imath/\sqrt{7}\sinh\varepsilon[2(\alpha_2 - \alpha_5) + \alpha_3 - \alpha_4]. && (2.117)
\end{aligned}$$

These roots reproduce the B_3-Cartan matrix, but it is not possible to express the imaginary part in terms of the undeformed B_3-roots. As expected from section 2.4.2, it is therefore impossible to find a three dimensional deformation matrix of the type (2.2). When identifying the undeformed A_6-roots related by the involution (2.29) according to $\alpha_1 \leftrightarrow \alpha_6$, $\alpha_2 \leftrightarrow \alpha_5$ and $\alpha_3 \leftrightarrow \alpha_4$, the deformed B_3-roots will all become real.

$F_4 \hookrightarrow E_6$ Having found some new solutions for a case which could not be solved previously, let us see next how some solutions we have found are related to each other through the folding procedure. In analogy to the undeformed case we may define the deformed F_4-roots in terms of the deformed E_6-roots as

$$\tilde{\beta}_1^{F_4} = \tilde{\alpha}_1^{E_6} + \tilde{\alpha}_6^{E_6}, \qquad \tilde{\beta}_2^{F_4} = \tilde{\alpha}_3^{E_6} + \tilde{\alpha}_5^{E_6}, \qquad \tilde{\beta}_3^{F_4} = \tilde{\alpha}_4^{E_6} \quad \text{and} \quad \tilde{\beta}_4^{F_4} = \tilde{\alpha}_3^{E_6}. \tag{2.118}$$

This means the F_4-deformation matrix is constructed as

$$\theta_\varepsilon^{F_4} = \begin{pmatrix} \frac{\theta_{11}^{E_6}+\theta_{61}^{E_6}+\theta_{16}^{E_6}+\theta_{66}^{E_6}}{2} & \frac{\theta_{13}^{E_6}+\theta_{63}^{E_6}+\theta_{15}^{E_6}+\theta_{65}^{E_6}}{2} & \theta_{14}^{E_6}+\theta_{64}^{E_6} & \theta_{12}^{E_6}+\theta_{62}^{E_6} \\ \frac{\theta_{31}^{E_6}+\theta_{51}^{E_6}+\theta_{36}^{E_6}+\theta_{56}^{E_6}}{2} & \frac{\theta_{33}^{E_6}+\theta_{53}^{E_6}+\theta_{35}^{E_6}+\theta_{55}^{E_6}}{2} & \theta_{34}^{E_6}+\theta_{54}^{E_6} & \theta_{32}^{E_6}+\theta_{52}^{E_6} \\ \frac{\theta_{41}^{E_6}+\theta_{46}^{E_6}}{2} & \frac{\theta_{43}^{E_6}+\theta_{45}^{E_6}}{2} & \theta_{44}^{E_6} & \theta_{42}^{E_6} \\ \frac{\theta_{21}^{E_6}+\theta_{26}^{E_6}}{2} & \frac{\theta_{23}^{E_6}+\theta_{25}^{E_6}}{2} & \theta_{24}^{E_6} & \theta_{22}^{E_6} \end{pmatrix}. \tag{2.119}$$

In this reduction the two inequivalent deformed E_6-root systems (2.101) and (2.102) produce the same solution for F_4

$$\theta_\varepsilon^{F_4} = \begin{pmatrix} r_0 - \imath r_k & -2\imath r_k & -4\imath r_k & -4\imath r_k \\ 2\imath r_k & r_0 + 5\imath r_k & 8\imath r_k & 4\imath r_k \\ -2\imath r_k & -4\imath r_k & r_0 - 5\imath r_k & -2\imath r_k \\ 2\imath r_k & 2\imath r_k & 2\imath r_k & r_0 + \imath r_k \end{pmatrix}. \tag{2.120}$$

This solution corresponds to a special solution we found in the context of F_4, namely (2.106) with $r_4 = r_6 = 0$.

Using the same identification between the F_4 and E_6 roots as in (2.118), we obtain from the solution based on the deformation of the longest element (2.103)

$$\tilde{\beta}_1^{F_4} = \alpha_1^{E_6} + \alpha_6^{E_6}, \tag{2.121}$$

$$\tilde{\beta}_2^{F_4} = (\cosh\varepsilon - \imath\sinh\varepsilon)\alpha_3^{E_6} + (\cosh\varepsilon + \imath\sinh\varepsilon)\alpha_5^{E_6}, \tag{2.122}$$

$$\tilde{\beta}_3^{F_4} = \frac{1}{2}(1 - \cosh\varepsilon + \imath\sinh\varepsilon)\alpha_3^{E_6} + \alpha_4^{E_6} + \frac{1}{2}(1 - \cosh\varepsilon - \imath\sinh\varepsilon)\alpha_5^{E_6} \tag{2.123}$$

$$\tilde{\beta}_4^{F_4} = \cosh\varepsilon\alpha_3^{E_6} + \imath\sinh\varepsilon\alpha_5^{E_6}. \tag{2.124}$$

These roots reproduce the F_4-Cartan matrix, but it is not possible to express them in terms of the undeformed F_4-roots. This reflects the fact that the longest elements acts trivially in this case and therefore also no nontrivial deformation of this involution exists.

3. Antilinear deformations of Calogero models

We have constructed a deformation map δ which replaces each root α by a deformed counterpart $\tilde{\alpha}$ as specified above. We will now employ this construction in the context of a concrete physical model and replace the set of n-dynamical variables $q = \{q_1, \ldots, q_n\}$ and their conjugate momenta $p = \{p_1, \ldots, p_n\}$ by means of this deformation map $\delta : (q, p) \to (\tilde{q}, \tilde{p})$.

3.1 The l=0 wavefunctions and eigenenergies in the undeformed case

Let us first generalize Calogero's construction [30] for the solution of the $l = 0$ wavefunction to generic Coxeter groups $\mathcal{W}$. We consider the generalized[1] Calogero Hamiltonian

$$\mathcal{H}_C(p, q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2}, \tag{3.1}$$

with g_α being real coupling constants, which for the time being may be different for each positive root $\alpha \in \Delta^+$ associated to any Coxeter group $\mathcal{W}$. Generalizing [30] we define now the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot q) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2, \tag{3.2}$$

where $\hat{h}$ denotes the dual Coxeter number and t_ℓ is the ℓ-th symmetrizer of the incidence matrix I defined through the relation $I_{ij}t_j = t_i I_{ij}$, see appendix B for some concrete values. We assume next that the wavefunction can be separated in terms of these variables in the form

$$\psi(q) \to \psi(z, r) = z^{\kappa+1/2}\varphi(r), \tag{3.3}$$

[1] In the sense of being not dependent on a specific representation of the roots and a particular Coxeter group.

with κ being an undetermined constant for the moment. Using this Ansatz we try to solve the n-body Schrödinger equation in position space $\mathcal{H}_C\psi(q) = E\psi(q)$ with $p^2 = -\sum_{i=1}^n \partial_{q_i}^2$. Changing variables for the Laplace operator then yields

$$\left\{-\frac{1}{2}\sum_{i=1}^{n}\left[\left(\kappa^2-\frac{1}{4}\right)\frac{1}{z^2}\left(\frac{\partial z}{\partial q_i}\right)^2+\left(\kappa+\frac{1}{2}\right)\frac{1}{z}\left(\frac{\partial^2 z}{\partial q_i^2}+2\frac{\partial z}{\partial q_i}\frac{\partial r}{\partial q_i}\frac{\partial}{\partial r}\right)+\frac{\partial^2 r}{\partial q_i^2}\frac{\partial^2}{\partial r^2}\right.\right. \tag{3.4}$$
$$\left.\left.+\left(\frac{\partial r}{\partial q_i}\right)^2\frac{\partial}{\partial r}\right]+\frac{\omega^2}{4}\hat{h}t_\ell r^2+\sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2}-E\right\}\varphi(r)=0.$$

Taking now $g_\alpha = g\alpha^2/2$, i.e. having the same coupling constant for all short and all long roots, and using the identities (A.7)-(A.11) from appendix A this reduces to

$$\left\{-\frac{1}{2}\left[\frac{\partial^2}{\partial r^2}+\left[\left(\kappa+\frac{1}{2}\right)h\ell+(\ell+1)\right]\frac{1}{r}\frac{\partial}{\partial r}\right]+\frac{\omega^2}{4}\hat{h}t_\ell r^2\right\}\varphi(r)=E\varphi(r). \tag{3.5}$$

The key feature is that due to the identity (A.7) the first term in (3.4) combines with part of the potential term to

$$\left[\frac{g}{2}-\frac{1}{2}\left(\kappa^2-\frac{1}{4}\right)\right]\sum_{\alpha\in\Delta^+}\frac{\alpha^2}{(\alpha\cdot q)^2}. \tag{3.6}$$

This term vanishes when choosing the free parameter κ to $\kappa = \pm 1/2\sqrt{1+4g}$. The positive solution is the only physical acceptable one, as we would obtain singularities in (3.3) and therefore a nonnormalizable wavefunction otherwise.

The equation (3.5) is a second order differential equation which may be solved by standard methods. Imposing as usual the physical constraint that the wavefunction vanishes at infinity, the energy quantizes to

$$E_n = \frac{1}{4}\left[\left(2+h+h\sqrt{1+4g}\right)l+8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega \tag{3.7}$$

with corresponding wavefunctions

$$\varphi_n(r) = c_n\exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}}\frac{\omega}{2}r^2\right)L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right). \tag{3.8}$$

Here $L_n^a(x)$ denotes the generalized Laguerre polynomial, c_n is a normalization constant and $a = \left(2+h+h\sqrt{1+4g}\right)l/4-1$.

A key feature of the model is that the last term in the potential in (3.1) becomes singular whenever $q_i = q_j$ for any $i,j \in \{1,2,\ldots,n\}$. This means that the wavefunction is vanishing at these points and we may encounter nontrivial phases for any two particle interchange. In fact, as the variable z defined in (3.2) is antisymmetric and r is symmetric in all variables it is easy to see that the associated particles give rise to anyonic exchange factors

$$\psi(q_1,\ldots,q_i,q_j,\ldots q_n) = e^{\imath\pi s}\psi(q_1,\ldots,q_j,q_i,\ldots q_n), \quad \text{for } 1\leq i,j\leq n, \tag{3.9}$$

with

$$s = \frac{1}{2} + \frac{1}{2}\sqrt{1+4g}. \tag{3.10}$$

This property of the model will change in the deformed case.

3.2 The l=0 wavefunctions and eigenenergies in the deformed case

Now we consider the antilinear deformation of the Calogero Hamiltonian $\delta : \mathcal{H}_C(\alpha) \to \mathcal{H}_{adC}(\tilde{\alpha})$

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\tilde{\alpha}\in\tilde{\Delta}^+}(\tilde{\alpha}\cdot q)^2 + \sum_{\tilde{\alpha}\in\Delta^+}\frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}. \tag{3.11}$$

In analogy to the deformed case we attempt to solve this model by a similar reparameterization as (3.2), i.e. defining the variables

$$\tilde{z} := \prod_{\tilde{\alpha}\in\tilde{\Delta}^+}(\tilde{\alpha}\cdot q) \qquad \text{and} \qquad \tilde{r}^2 := \frac{1}{\hat{h}t_\ell}\sum_{\tilde{\alpha}\in\tilde{\Delta}^+}(\tilde{\alpha}\cdot q)^2, \tag{3.12}$$

and separating the wavefunction as

$$\psi(q) \to \psi(\tilde{z},\tilde{r}) = \tilde{z}^s\varphi(\tilde{r}). \tag{3.13}$$

As a consequence of our construction for the deformed roots for which we demanded that inner products are preserved, we find that $\tilde{r} = r$. Furthermore, we observe that due to this fact the relations (A.2) and (A.4) also hold when replacing α by $\tilde{\alpha}$ and consequently the solution procedure for the eigenvalue equation does not change. Therefore we obtain

$$\psi(q) = \psi(\tilde{z},r) = \tilde{z}^s\varphi_n(r) \tag{3.14}$$

as solution with $\varphi_n(r)$ given in (3.8) and unchanged energy eigenvalues (3.7). When generalizing the ansatz (3.13) to take also values for $l \neq 0$ into account the energy eigenvalues will, however, change, as was demonstrated in [18] for A_2 and G_2. The main difference between the deformed and undeformed case for the solution provided here is the occurrence of the variable $\tilde{z}$ instead of z. As a consequence the wavefunction (3.14) no longer vanishes when two q_is values coincide, which in turn is a reflection of the fact that all singularities resulting from a two-particle exchange have been regularized by means of the deformation. However, we still encounter singularities in the potential when all n values for the q_is coincide. The wavefunction vanishes in this case and we obtain nontrivial statistics exchange factors.

Let us see in detail for some concrete models how to obtain nontrivial anyonic exchange factors for an n-particle scattering process.

3.2.1 The deformed A_2-model

The potential in (3.11) and the variable $\tilde{z}$ in (3.12) are computed from the inner products of all 3 roots in $\tilde{\Delta}^+_{A_2}$ with the vector q. Using the standard three dimensional representation

for the simple A_2-roots $\alpha_1 = \{1, -1, 0\}$ and $\alpha_2 = \{0, 1, -1\}$, we find with (2.47) and (2.48)

$$\tilde{\alpha}_1 \cdot q = q_{12} \cosh \varepsilon - \frac{\imath}{\sqrt{3}} (q_{13} + q_{23}) \sinh \varepsilon, \tag{3.15}$$

$$\tilde{\alpha}_2 \cdot q = q_{23} \cosh \varepsilon - \frac{\imath}{\sqrt{3}} (q_{21} + q_{31}) \sinh \varepsilon, \tag{3.16}$$

$$(\tilde{\alpha}_1 + \tilde{\alpha}_2) \cdot q = q_{13} \cosh \varepsilon + \frac{\imath}{\sqrt{3}} (q_{12} + q_{32}) \sinh \varepsilon. \tag{3.17}$$

For convenience we introduced the notation $q_{ij} := q_i - q_j$. The new feature of these models is that the last term in the potential (3.11) resulting from these products is no longer singular when the position of two particles coincides. It is easy to see that the $\mathcal{PT}$-symmetry constructed for the $\tilde{\alpha}$ may be realized alternatively in the dual space, that is on the level of the dynamical variables

$$\sigma_-^\varepsilon : \quad \tilde{\alpha}_1 \leftrightarrow -\tilde{\alpha}_1,\ \tilde{\alpha}_2 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \Leftrightarrow \quad q_1 \leftrightarrow q_2,\ q_3 \leftrightarrow q_3,\ \imath \to -\imath, \tag{3.18}$$

$$\sigma_+^\varepsilon : \quad \tilde{\alpha}_2 \leftrightarrow -\tilde{\alpha}_2,\ \tilde{\alpha}_1 \leftrightarrow \tilde{\alpha}_1 + \tilde{\alpha}_2 \quad \Leftrightarrow \quad q_2 \leftrightarrow q_3,\ q_1 \leftrightarrow q_1,\ \imath \to -\imath. \tag{3.19}$$

A crucial difference to the undeformed case is that $\tilde{z}$ will, unlike z, not vanish in the two particle scattering process when two qs coincide. In fact in that case $\tilde{z}$ will be purely imaginary as follows directly from the $\mathcal{PT}$-symmetry

$$\sigma_-^\varepsilon \tilde{z}(q_1, q_2, q_3) = \tilde{z}^*(q_2, q_1, q_3) = -\tilde{z}(q_1, q_2, q_3) \quad \Rightarrow \quad \tilde{z}(q_1, q_1, q_3) \in \imath\mathbb{R}, \tag{3.20}$$

$$\sigma_+^\varepsilon \tilde{z}(q_1, q_2, q_3) = \tilde{z}^*(q_1, q_3, q_2) = -\tilde{z}(q_1, q_2, q_3) \quad \Rightarrow \quad \tilde{z}(q_1, q_3, q_3) \in \imath\mathbb{R}. \tag{3.21}$$

The remaining possibility $\tilde{z}(q_1, q_2, q_1) \in \imath\mathbb{R}$ follows from the previous cases together with the cyclic property $\tilde{z}(q_1, q_2, q_3) = \tilde{z}(q_2, q_3, q_1)$, which in turn results when combining (3.20) and (3.21). Under these circumstances a new symmetry arises

$$\alpha_1 = 0,\ \alpha_2 \to -\alpha_2 \quad \Leftrightarrow \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\ \tilde{\alpha}_2 \to -\tilde{\alpha}_2 \quad \Leftrightarrow \quad q_1 = q_2,\ q_2 \leftrightarrow q_3, \tag{3.22}$$

leading to $\tilde{z}(q_2, q_2, q_3) = -\tilde{z}(q_3, q_3, q_2)$. By (3.3) this means

$$\psi(q_2, q_2, q_3) = e^{\imath\pi s} \psi(q_3, q_3, q_2) \tag{3.23}$$

with s given in (3.10). Hence we obtain a nontrivial exchange factor in the three-particle scattering process when particle 1 and 2 have the same position and are simultaneously scattered with particle 3.
Similarly we observe

$$\alpha_2 = 0,\ \alpha_1 \to -\alpha_1 \quad \Leftrightarrow \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\ \tilde{\alpha}_2 \to -\tilde{\alpha}_2 \quad \Leftrightarrow \quad q_2 = q_3,\ q_1 \leftrightarrow q_2, \tag{3.24}$$

leading to $\tilde{z}(q_1, q_2, q_2) = -\tilde{z}(q_2, q_2, q_1)$ and therefore

$$\psi(q_1, q_2, q_2) = e^{\imath\pi s} \psi(q_2, q_2, q_1). \tag{3.25}$$

Now a nontrivial exchange factor emerges in the three-particle scattering process when particle 2 and 3 have the same position and are simultaneously scattered with particle 1. We depict various possibilities in figure 2.

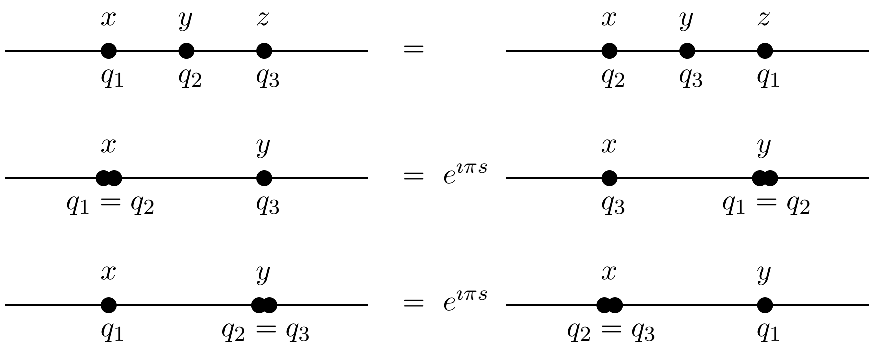

Figure 2: Anyonic exchange factors for the 3-particle scattering in the A_2-model.

Notice that the first case in figure 2, leading to a bosonic exchange possesses an analogue in the undeformed case. This process can be viewed in two alternative ways, either corresponding to two consecutive two particle exchanges, i.e. $1 \leftrightarrow 2$ and subsequently $1 \leftrightarrow 3$, or equivalently to a simultaneous three particle scattering process that is the ordering 123 goes to 231 in one scattering event. This is the typical factorization of an n-particle scattering process into a sequence of two-particle scatterings encountered in integrable models, see e.g. [31]. In fact, as this feature is so central it is often used synonymously with integrability. In our deformed model we encounter new possibilities, namely that a compound particle can exist in the first place and then also scatter with a single particle; giving rise to anyonic exchange factors in this case.

3.2.2 Deformed A_3-models

Based on $\mathcal{PT}$-symmetrically deformed Coxeter group factors In this case the potential and $\tilde{z}$ are computed from the inner products of all 6 roots in $\tilde{\Delta}^+_{A_3}$ with q. Taking the simple roots in the standard four dimensional representation $\alpha_1 = \{1,-1,0,0\}$, $\alpha_2 = \{0,1,-1,0\}$, $\alpha_3 = \{0,0,1,-1\}$, we evaluate with (2.53) and (2.59)

$$\tilde{\alpha}_1 \cdot q = q_{43} + \cosh\varepsilon (q_{12}+q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13}+q_{24}), \tag{3.26}$$

$$\tilde{\alpha}_2 \cdot q = q_{23}(2\cosh\varepsilon - 1) + \imath 2\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{14}, \tag{3.27}$$

$$\tilde{\alpha}_3 \cdot q = q_{21} + \cosh\varepsilon (q_{12}+q_{34}) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{13}+q_{24}), \tag{3.28}$$

$$\tilde{\alpha}_4 \cdot q = q_{42} + \cosh\varepsilon (q_{13}+q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12}+q_{34}), \tag{3.29}$$

$$\tilde{\alpha}_5 \cdot q = q_{31} + \cosh\varepsilon (q_{13}+q_{24}) + \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2}(q_{12}+q_{34}), \tag{3.30}$$

$$\tilde{\alpha}_6 \cdot q = q_{14}(2\cosh\varepsilon - 1) - \imath\sqrt{2\cosh\varepsilon}\sinh\frac{\varepsilon}{2} q_{23}. \tag{3.31}$$

Once again the last term in the potential (3.11) resulting from these products is no longer singular in any two particle exchange. However, in this case it could become singular in two simultaneous two-particle scattering processes, e.g. $q_{14} = q_{23} = 0$. Once again we may

realize the $\mathcal{PT}$-symmetry constructed for the $\tilde{\alpha}$

$$\sigma_-^\varepsilon: \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\ \tilde{\alpha}_2 \to \tilde{\alpha}_6,\ \tilde{\alpha}_3 \to -\tilde{\alpha}_3,\ \tilde{\alpha}_4 \to \tilde{\alpha}_5,\ \tilde{\alpha}_5 \to \tilde{\alpha}_4,\ \tilde{\alpha}_6 \to \tilde{\alpha}_2, \tag{3.32}$$

$$\sigma_+^\varepsilon: \quad \tilde{\alpha}_1 \to \tilde{\alpha}_4,\ \tilde{\alpha}_2 \to -\tilde{\alpha}_2,\ \tilde{\alpha}_3 \to \tilde{\alpha}_5,\ \tilde{\alpha}_4 \to \tilde{\alpha}_1,\ \tilde{\alpha}_5 \to \tilde{\alpha}_3,\ \tilde{\alpha}_6 \to \tilde{\alpha}_6, \tag{3.33}$$

also in the dual space

$$\sigma_-^\varepsilon: \quad q_1 \to q_2,\ q_2 \to q_1,\ q_3 \to q_4,\ q_4 \to q_3,\ \imath \to -\imath, \tag{3.34}$$

$$\sigma_+^\varepsilon: \quad q_1 \to q_1,\ q_2 \to q_3,\ q_3 \to q_2,\ q_4 \to q_4,\ \imath \to -\imath. \tag{3.35}$$

As in the A_2-case $\tilde{z}$ will not vanish when two qs coincide, but once again we may pick up nontrivial exchange factors when involving all particles in the model in the scattering process. We observe

$$\sigma_-^\varepsilon \tilde{z}(q_1,q_2,q_3,q_4) = \tilde{z}^*(q_2,q_1,q_4,q_3) = \tilde{z}(q_1,q_2,q_3,q_4), \tag{3.36}$$

$$\sigma_+^\varepsilon \tilde{z}(q_1,q_2,q_3,q_4) = \tilde{z}^*(q_1,q_3,q_2,q_4) = -\tilde{z}(q_1,q_2,q_3,q_4). \tag{3.37}$$

Combining (3.36) and (3.37) then yields

$$\tilde{z}(q_1,q_2,q_3,q_4) = -\tilde{z}(q_2,q_4,q_1,q_3), \tag{3.38}$$

and therefore we will encounter nontrivial exchange factors in a 4-particle scattering process

$$\psi(q_1,q_2,q_3,q_4) = e^{\imath\pi s}\psi(q_2,q_4,q_1,q_3). \tag{3.39}$$

We depict various possibilities in figure 3.

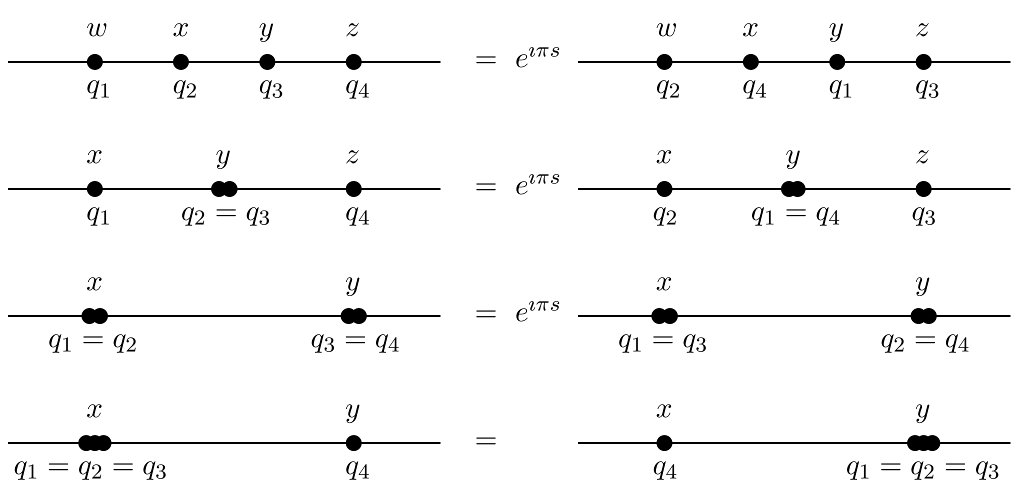

Figure 3: Anyonic exchange factors for the 4-particle scattering in the A_3-model.

As in the previous case we encounter several possibilities which have no counterpart in the undeformed case.

Based on $\mathcal{CT}$-symmetrically deformed longest element We keep now the representation for the simple roots, but use the construction for the deformed roots as provided in the second part of section 2.4.1. The potential is obtained again by computing

$$\tilde{\alpha}_1 \cdot q = \cosh\varepsilon q_{12} + \imath \sinh\varepsilon q_{34}, \tag{3.40}$$
$$\tilde{\alpha}_2 \cdot q = \cosh^2\frac{\varepsilon}{2} q_{23} - \sinh^2\frac{\varepsilon}{2} q_{14} + \frac{\imath}{2}\sinh\varepsilon(q_{12} + q_{43}), \tag{3.41}$$
$$\tilde{\alpha}_3 \cdot q = \cosh\varepsilon q_{34} + \imath \sinh\varepsilon q_{21}, \tag{3.42}$$
$$\tilde{\alpha}_4 \cdot q = \cosh\varepsilon q_{13} - \imath \sinh\varepsilon q_{24}, \tag{3.43}$$
$$\tilde{\alpha}_5 \cdot q = \cosh\varepsilon q_{24} + \imath \sinh\varepsilon q_{13}, \tag{3.44}$$
$$\tilde{\alpha}_6 \cdot q = \cosh^2\frac{\varepsilon}{2} q_{14} + \sinh^2\frac{\varepsilon}{2} q_{23} + \frac{\imath}{2}\sinh\varepsilon(q_{21} + q_{34}). \tag{3.45}$$

Clearly the potential is different from the one resulting from (3.26)-(3.31). Despite the fact that it is a simpler potential, it can not be solved analogously to the previous case since the crucial relations (A.1)-(A.4) no longer hold.

3.2.3 The deformed F_4-model

In order to unravel any features which might differ in the non-simply laced case, which is usually the case, we also present her one example for such a model. To allow a direct comparison with the previous 4-particle case, we have selected F_4. The positive root space $\tilde{\Delta}^+_{F_4}$ contains now 24 root. Taking the simple roots in the standard four dimensional representation $\alpha_1 = \{0, 1, -1, 0\}$, $\alpha_2 = \{0, 0, 1, -1\}$, $\alpha_3 = \{0, 0, 0, 1\}$ and $\alpha_4 = \{1/2, -1/2, -1/2, -1/2\}$ we compute the following factorization for $\tilde{z}$, with each factor corresponding to one of the 24 products $\tilde{\alpha}_i \cdot q$

$$
\begin{aligned}
&(q_1 \cosh\varepsilon + \imath \sinh\varepsilon q_4)(q_2 \cosh\varepsilon - \imath \sinh\varepsilon q_3)(q_3 \cosh\varepsilon + \imath \sinh\varepsilon q_2)(q_4 \cosh\varepsilon - \imath \sinh\varepsilon q_1) \\
&\times (q_{12} \cosh\varepsilon + \imath \sinh\varepsilon \hat{q}_{34})(q_{14} \cosh\varepsilon + \imath \sinh\varepsilon \hat{q}_{14})(q_{34} \cosh\varepsilon + \imath \sinh\varepsilon \hat{q}_{12}) \\
&\times (q_{23} \cosh\varepsilon - \imath \sinh\varepsilon \hat{q}_{23})(\hat{q}_{13} \cosh\varepsilon + \imath \sinh\varepsilon \hat{q}_{24})(\hat{q}_{24} \cosh\varepsilon - \imath \sinh\varepsilon \hat{q}_{13}) \\
&\times (\hat{q}_{34} \cosh\varepsilon - \imath \sinh\varepsilon q_{12})(\hat{q}_{23} \cosh\varepsilon + \imath \sinh\varepsilon \hat{q}_{23})(\hat{q}_{12} \cosh\varepsilon - \imath \sinh\varepsilon \hat{q}_{34}) \\
&\times (\hat{q}_{14} \cosh\varepsilon - \imath \sinh\varepsilon q_{14})(q_{24} \cosh\varepsilon + \imath \sinh\varepsilon q_{13})(q_{13} \cosh\varepsilon - \imath \sinh\varepsilon q_{24}) \\
&\times \left[\frac{\hat{q}_{12} + \hat{q}_{34}}{2}\cosh\varepsilon - \frac{\imath}{2}\sinh\varepsilon(q_{12} + q_{34})\right]\left[\frac{\hat{q}_{12} - q_{34}}{2}\cosh\varepsilon - \frac{\imath}{2}\sinh\varepsilon(\hat{q}_{12} + q_{34})\right] \\
&\times \left[\frac{q_{12} - \hat{q}_{34}}{2}\cosh\varepsilon + \frac{\imath}{2}\sinh\varepsilon(q_{12} + \hat{q}_{34})\right]\left[\frac{\hat{q}_{12} - \hat{q}_{34}}{2}\cosh\varepsilon + \frac{\imath}{2}\sinh\varepsilon(q_{12} - q_{34})\right] \\
&\times \left[\frac{q_{12} + \hat{q}_{34}}{2}\cosh\varepsilon - \frac{\imath}{2}\sinh\varepsilon(q_{12} - \hat{q}_{34})\right]\left[\frac{q_{12} - q_{34}}{2}\cosh\varepsilon - \frac{\imath}{2}\sinh\varepsilon(\hat{q}_{12} - \hat{q}_{34})\right] \\
&\times \left[\frac{q_{12} + q_{34}}{2}\cosh\varepsilon + \frac{\imath}{2}\sinh\varepsilon(\hat{q}_{12} + \hat{q}_{34})\right]\left[\frac{q_{12} + q_{34}}{2}\cosh\varepsilon + \frac{\imath}{2}\sinh\varepsilon(\hat{q}_{12} - q_{34})\right],
\end{aligned}
$$

where we used the abbreviation $\hat{q}_{ij} := q_i + q_j$. Once again, several singularities have disappeared through the deformation. The $\mathcal{PT}$-symmetry constructed for the simple deformed

roots $\tilde{\alpha}$

$$\sigma_-^\varepsilon: \quad \tilde{\alpha}_1 \to -\tilde{\alpha}_1,\, \tilde{\alpha}_2 \to \tilde{\alpha}_1 + \tilde{\alpha}_2 + 2\tilde{\alpha}_3,\, \tilde{\alpha}_3 \to -\tilde{\alpha}_3,\, \tilde{\alpha}_4 \to \tilde{\alpha}_3 + \tilde{\alpha}_4, \tag{3.46}$$

$$\sigma_+^\varepsilon: \quad \tilde{\alpha}_1 \to \tilde{\alpha}_1 + \tilde{\alpha}_2,\, \tilde{\alpha}_2 \to -\tilde{\alpha}_2,\, \tilde{\alpha}_3 \to \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4,\, \tilde{\alpha}_4 \to -\tilde{\alpha}_4, \tag{3.47}$$

is now realized in the dual space as

$$\sigma_-^\varepsilon: \quad q_1 \to q_1,\, q_2 \to q_3,\, q_3 \to q_2,\, q_4 \to -q_4,\, \imath \to -\imath, \tag{3.48}$$

$$\sigma_+^\varepsilon: \quad q_1 \to \frac{1}{2}(q_1+q_2+q_3+q_4),\, q_2 \to \frac{1}{2}(q_1+q_2-q_3-q_4), \tag{3.49}$$

$$q_3 \to \frac{1}{2}(q_1-q_2-q_3+q_4),\, q_4 \to \frac{1}{2}(q_1-q_2+q_3-q_4),\, \imath \to -\imath. \tag{3.50}$$

Now we observe

$$\sigma_-^\varepsilon \tilde{z}(q_1,q_2,q_3,q_4) = \tilde{z}^*(q_1,q_3,q_2,-q_4) = \tilde{z}(q_1,q_2,q_3,q_4), \tag{3.51}$$

$$\sigma_+^\varepsilon \tilde{z}(q_1,q_2,q_3,q_4) = \tilde{z}^* \left[\frac{\hat{q}_{12}+\hat{q}_{34}}{2}, \frac{\hat{q}_{12}-\hat{q}_{34}}{2}, \frac{q_{12}-q_{34}}{2}, \frac{q_{12}+q_{34}}{2} \right] = \tilde{z}(q_1,q_2,q_3,q_4). \tag{3.52}$$

A consequence of this we find the symmetry

$$\psi(q_1,q_2,q_3,q_4) = \psi(\frac{\hat{q}_{13}+q_{24}}{2}, \frac{\hat{q}_{13}-q_{34}}{2}, \frac{q_{13}-\hat{q}_{24}}{2}, \frac{q_{13}+\hat{q}_{24}}{2}), \tag{3.53}$$

which gives rise to new possibilities neither encountered in the undeformed case nor in the deformed A_3-case.

4. Conclusions

As a particular element in the Coxeter group we have selected the involutions $\sigma_\pm$, $\omega_0 \in \mathcal{W}$ and deformed them in an antilinear manner. For each construction we have set up sets of constraining equations (2.15) and (2.36) for the deformation matrix θ_ε and solved them case-by-case. This matrix then yields via (2.2) the deformed simple roots and with the help of the deformed Coxeter element σ^ε we have constructed the remaining nonsimple roots and thus the entire deformed root space $\tilde{\Delta}(\varepsilon)$. Depending on the type of construction this space remains invariant under the action of the new elements $\sigma_\pm^\varepsilon$, ω_0^ε, i.e. an antilinear transformation, and in the former case also under the action of σ^ε. The construction based on the deformation of ω_0 is more restrictive from the very onset as it can only be applied to A_ℓ, $D_{2\ell}$ and E_6. In addition, the resulting deformed root space is only invariant under the action of σ^ε when the construction coincides with the one based on the deformation of the $\sigma_\pm$. The latter construction is more general as it can be applied to all Coxeter groups, albeit it does not always lead to nontrivial solutions. We have demonstrated that for the cases for which one can only find real solutions one may still find complex solutions of different type by means of the folding procedure.

Clearly there are various open issues with regard to the mathematical framework. We have for instance not constructed in all cases the most general deformation possible, even for the constraints we have provided. Besides the closed generic formulae we found for some

infinite series it might be possible to construct them also for the missing ones. Furthermore, it would be interesting to relax some of the constraints we impose on the deformation matrix and construct new types of solutions, especially in those cases for which we showed that no complex solutions exist. Finally it would also be interesting to investigate the possibility to deform different types of involutions besides the $\sigma_{\pm}$ and ω_0 presented here.

We argued that the deformed root systems may be used to define new types of physical models, whose formulation is based on root systems, such as Toda lattice theories, affine or conformal Toda field theories, Calogero-Moser-Sutherland models etc. By construction the Hamiltonians related to all these models will be invariant under some antilinear transformation, $\sigma_{\pm}^{\varepsilon}$ or ω_0^{ε}, and therefore have a strong likelihood to be physically meaningful with real spectra. We have exploited this possibility in some detail for Calogero models. We have constructed specific solutions to the Schrödinger equation leading to the same real energy spectrum as in the undeformed case, but to different wavefunctions. A particular interesting new feature of the new models is that they give rise to anyonic exchange factors of various types of particle exchanges which have no analogue in the undeformed case. Our solution procedure relies entirely on the validity of the identities presented in appendix A, for which we have presented strong evidence on various case-by-case studies. For completeness it would be desirable to have some generic proofs for them. Most interesting from a physical point of view would be to find further solutions to the Schrödinger equation besides the $l = 0$ ones. As in the G_2 and A_2 case this will most likely give rise to a different energy spectrum similarly to the undeformed case. Related to this issue is the interesting question of how the Hermitian counterpart or possibly counterparts to $\mathcal{H}_{adC}(p, q)$ would look like. Naturally one would also like to answer the question of whether the deformed models are still integrable, which is especially interesting in the light of the comments made at the end of section 3.2.1.

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A. Identities

We assemble here the crucial identities for the derivation of the radial part of the Schrödinger equation (3.5). Underlying are the generic relations which only involve roots and the dynamical variables $q = \{q_1, \ldots, q_n\}$

$$\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} = \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \tag{A.1}$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} = \frac{\hat{h}h\ell}{2} t_\ell, \tag{A.2}$$

$$\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) = \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2, \tag{A.3}$$

$$\sum_{\alpha\in\Delta^+} \alpha^2 = \ell\hat{h}t_\ell. \tag{A.4}$$

At present we do not have a generic proof for these relations. A large amount of evidence on a case-by-case basis for the first identity was already provided in [15], albeit no case independent generic proof. Here we have verified (A.2) and (A.3) for a substantial number of Coxeter groups. Denoting by n_s, α_s^2 and n_l, α_l^2 the number and length of the short and long roots, respectively, (A.4) follows from

$$\sum_{\alpha\in\Delta^+} \alpha^2 = \frac{n_s}{2}\alpha_s^2 + \frac{n_l}{2}\alpha_l^2 = \frac{\alpha_l^2}{2}\left(n_s\frac{\alpha_s^2}{\alpha_l^2} + n_l\right) = \ell\hat{h}t_\ell, \tag{A.5}$$

where we used $n_s\alpha_s^2/\alpha_l^2 + n_l = \ell\hat{h}$, which can be found for instance in [32] and $\alpha_l^2 = 2t_\ell$.

Accepting the relations (A.1)-(A.4) the identities involving derivatives of r and z are easily derived. From (3.2) follows

$$\frac{\partial z}{\partial q_i} = z\sum_{\alpha\in\Delta^+} \frac{\alpha^i}{(\alpha\cdot q)} \qquad \text{and} \qquad \frac{\partial r}{\partial q_i} = \frac{1}{r\hat{h}t_\ell}\sum_{\alpha\in\Delta^+} (\alpha\cdot q)\alpha^i. \tag{A.6}$$

Multiplying them and summing over the dynamical variables gives

$$\sum_{i=1}^n \left(\frac{\partial z}{\partial q_i}\right)^2 = z^2 \sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} = z^2\sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \tag{A.7}$$

$$\sum_{i=1}^n \frac{\partial z}{\partial q_i}\frac{\partial r}{\partial q_i} = \frac{z}{\hat{h}t_\ell r}\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} = \frac{h\ell}{2}\frac{z}{r}, \tag{A.8}$$

$$\sum_{i=1}^n \left(\frac{\partial r}{\partial q_i}\right)^2 = \frac{1}{r^2\hat{h}^2t_\ell^2}\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) = 1, \tag{A.9}$$

where we have used (A.1) in (A.7), (A.2) in (A.8) and (A.3) in (A.9). Furthermore we need the sums over the second order derivatives. From (A.6) we obtain with the help of

(A.1) and (A.2) the two relations

$$\sum_{i=1}^{n} \frac{\partial^2 z}{\partial q_i^2} = z \left(\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} - \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2} \right) = 0, \tag{A.10}$$

$$\sum_{i=1}^{n} \frac{\partial^2 r}{\partial q_i^2} = \frac{1}{r\hat{h}t_\ell} \sum_{\alpha\in\Delta^+} \alpha^2 - \frac{1}{r^3\hat{h}t_\ell} \sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) = \frac{\ell-1}{r}. \tag{A.11}$$

B. Case-by-case data

For convenience we present in this appendix some numerical data for individual Coxeter groups. We present the values for the Coxeter number h defined as the total number of roots divided by the rank, the order of the Coxeter element σ or $1 + \sum_{i=1}^{\ell} n_i$ when the highest root is expressed in terms of simple roots as $\sum_{i=1}^{\ell} n_i \alpha_i$. The dual Coxeter number is defined in the same way as the Coxeter number for the situation in which the arrows on the affine Diagram have been reversed. The exponents s_n are related to the eigenvalues of the Coxeter element as defined in (2.21) and t_ℓ is the ℓ-th symmetrizer of the incidence matrix I defined by means of the relation $I_{ij}t_j = t_i I_{ij}$.

$\mathcal{W}$	h	$\hat{h}$	s_n	t_ℓ
A_ℓ	$\ell+1$	$\ell+1$	$1,2,3,...,\ell$	1
B_ℓ	2ℓ	$2\ell-1$	$1,3,5,...,2\ell-1$	1
C_ℓ	2ℓ	$\ell+1$	$1,3,5,...,2\ell-1$	2
D_ℓ	$2\ell-2$	$2\ell-2$	$1,3,...,\ell-1,\ldots,2\ell-3$	1
E_6	12	12	$1,4,5,7,8,11$	1
E_7	18	18	$1,5,7,9,11,13,17$	1
E_8	30	30	$1,7,11,13,17,19,23,29$	1
F_4	12	9	$1,5,7,11$	1
G_2	6	4	$1,5$	3
H_3	10	10	$1,5,9$	1

Table 1: Coxeter number h, dual Coxeter number $\hat{h}$, exponents s_n and ℓth symmetrizer t_ℓ.

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